CHAPTER 5

Mathematical Induction and Binomial Theorem

Section-A JEE Advanced/ IIT-JEE

A Fill in the Blanks

1. The larger of $99^{50} + 100^{50}$ and 101^{50} is .... *(1982 - 2 Marks)*
2. The sum of the coefficients of the plynomial $(1 + x - 3x^2)^{2163}$ is .... *(1982 - 2 Marks)*
3. If $(1 + ax)^n = 1 + 8x + 24x^2 + ...$ then $a =$ ...... and $n =$ .... *(1983 - 2 Marks)*
4. Let n be positive integer. If the coefficients of 2nd, 3rd, and 4th terms in the expansion of $(1 + x)^n$ are in A.P., then the value of n is .... *(1994 - 2 Marks)*
5. The sum of the rational terms in the expansion of $(\sqrt{2} + 3^{1/5})^{10}$ is .... *(1997 - 2 Marks)*

C MCQs with One Correct Answer

1. Given positive integers $r > 1$, $n > 2$ and that the coefficient of $(3r)$th and $(r + 2)$th terms in the binomial expansion of $(1+x)^{2n}$ are equal . Then *(1983 - 1 Mark)*
   (a) $n = 2r$ (c) $n = 2r + 1$
   (c) $n = 3r$ (d) none of these
2. The coefficient of x^4 in $\left(\frac{x}{2} - \frac{3}{x^2}\right)^{10}$ is *(1983 - 1 Mark)*
   (a) $\frac{405}{256}$ (b) $\frac{504}{259}$
   (c) $\frac{450}{263}$ (d) none of these
3. The expression $\left(x + (x^3 - 1)^{\frac{1}{2}}\right)^5 + \left(x - (x^3 - 1)^{\frac{1}{2}}\right)^5$ is a polynomial of degree *(1992 - 2 Marks)*
   (a) 5 (b) 6 (c) 7 (d) 8
4. If in the expansion of $(1 + x)^m (1 - x)^n$, the coefficients of x and x^2 are 3 and – 6 respectively, then m is *(1999 - 2 Marks)*
   (a) 6 (b) 9 (c) 12 (d) 24
5. For $2 \le r \le n$, $\binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2} =$ *(2000S)*
   (a) $\binom{n+1}{r-1}$ (b) $2\binom{n+1}{r+1}$ (c) $2\binom{n+2}{r}$ (d) $\binom{n+2}{r}$
6. In the binomial expansion of $(a - b)^n$, $n \ge 5$, the sum of the 5th and 6th terms is zero. Then a/b equals *(2001S)*
   (a) $(n - 5)/6$ (b) $(n - 4)/5$
   (c) $5/(n - 4)$ (d) $6/(n - 5)$
7. The sum $\sum_{i=0}^{m} \binom{10}{i}\binom{20}{m-i}$, (where $\binom{p}{q} = 0$ if p < q) is maximum when m is *(2002S)*
   (a) 5 (b) 10 (c) 15 (d) 20
8. Coefficient of t^{24} in $(1 + t^2)^{12} (1 + t^{12}) (1 + t^{24})$ is *(2003S)*
   (a) ${}^{12}C_6 + 3$ (b) ${}^{12}C_6 + 1$ (c) ${}^{12}C_6$ (d) ${}^{12}C_6 + 2$
9. If ${}^{n-1}C_r = (k^2 - 3)\, {}^nC_{r+1}$, then $k \in$ *(2004S)*
   (a) $(-\infty, -2]$ (b) $[2, \infty)$ (c) $[-\sqrt{3}, \sqrt{3}]$ (d) $(\sqrt{3}, 2]$
10. The value of
   $$\binom{30}{0}\binom{30}{10} - \binom{30}{1}\binom{30}{11} + \binom{30}{2}\binom{30}{12} ..... + \binom{30}{20}\binom{30}{30}$$ is where $\binom{n}{r} = {}^nC_r$ *(2005S)*
   (a) $\binom{30}{10}$ (b) $\binom{30}{15}$ (c) $\binom{60}{30}$ (d) $\binom{31}{10}$
11. For r = 0, 1, …, 10, let A_r, B_r and C_r denote, respectively, the coefficient of x^r in the expansions of $(1 + x)^{10}$, $(1 + x)^{20}$ and $(1 + x)^{30}$. Then $\sum_{r=1}^{10} A_r(B_{10}B_r - C_{10}A_r)$ is equal to *(2010)*
   (a) $B_{10} - C_{10}$ (b) $A_{10}(B^2{}_{10}C_{10}A_{10})$
   (c) 0 (d) $C_{10} - B_{10}$
12. Coefficient of x^{11} in the expansion of $(1 + x^2)^4 (1 + x^3)^7 (1 + x^4)^{12}$ is *(JEE Adv. 2014)*
   (a) 1051 (b) 1106 (c) 1113 (d) 1120

D MCQs with One or More than One Correct

1. If C_r stands for ${}^nC_r$, then the sum of the series
   $$\frac{2\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}{n!}[C_0^2 - 2C_1^2 + 3C_2^2 - .... + (-1)^n (n + 1)C_n^2],$$
   where n is an even positive integer, is equal to *(1986 - 2 Marks)*

(a) 0 (b) $(-1)^{n/2}(n+1)$

(c) $(-1)^{n/2}(n+2)$ (d) $(-1)^n n$

(e) none of these.

2. If $a_n = \sum_{r=0}^{n} \frac{1}{{}^nC_r}$, then $\sum_{r=0}^{n} \frac{r}{{}^nC_r}$ equals *(1998 - 2 Marks)*

(a) $(n-1)a_n$ (b) na_n

(c) $\frac{1}{2}na_n$ (d) None of the above

E Subjective Problems

1. Given that *(1979)*
$C_1 + 2C_2x + 3C_3x^2 + \ldots\ldots\ldots\ldots + 2n\, C_{2n}x^{2n-1} = 2n(1+x)^{2n-1}$
where $C_r = \frac{(2n)!}{r!(2n-r)!}$ $r = 0, 1, 2, \ldots\ldots\ldots\ldots, 2n$
Prove that
$C_1^2 - 2C_2^2 + 3C_3^2 - \ldots\ldots\ldots\ldots - 2nC_{2n}^2 = (-1)^n n\, C_n.$

2. Prove that $7^{2n} + (2^{3n-3})(3^{n-1})$ is divisible by 25 for any natural number n. *(1982 - 5 Marks)*

3. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n$ then show that the sum of the products of the C_i's taken two at a time, represented by $\sum_{0\le i<j\le n}\sum C_iC_j$ is equal to $2^{2n-1} - \frac{(2n)!}{2(n!)^2}$ *(1983 - 3 Marks)*

4. Use mathematical Induction to prove : If n is any odd positive integer, then $n(n^2 - 1)$ is divisible by 24. *(1983 - 2 Marks)*

5. If p be a natural number then prove that $p^{n+1} + (p+1)^{2n-1}$ is divisible by $p^2 + p + 1$ for every positive integer n. *(1984 - 4 Marks)*

6. Given $s_n = 1 + q + q^2 + \ldots\ldots + q^n$;
$S_n = 1 + \frac{q+1}{2} + \left(\frac{q+1}{2}\right)^2 + \ldots\ldots + \left(\frac{q+1}{2}\right)^n, q \ne 1$ Prove that
${}^{n+1}C_1 + {}^{n+1}C_2s_1 + {}^{n+1}C_3s_2 + \ldots\ldots + {}^{n+1}C_ns_n = 2^nS_n$ *(1984 - 4 Marks)*

7. Use method of mathematical induction $2.7^n + 3.5^n - 5$ is divisible by 24 for all $n > 0$ *(1985 - 5 Marks)*

8. Prove by mathematical induction that – *(1987 - 3 Marks)*
$\frac{(2n)!}{2^{2n}(n!)^2} \le \frac{1}{(3n+1)^{1/2}}$ for all positive Integers n.

9. Let $R = (5\sqrt{5}+11)^{2n+1}$ and $f = R - [R]$, where [ ] denotes the greatest integer function. Prove that $Rf = 4^{2n+4}$. *(1988 - 5 Marks)*

10. Using mathematical induction, prove that *(1989 - 3 Marks)*
${}^mC_0\,{}^nC_k + {}^mC_1\,{}^nC_{k-1} + \ldots\ldots\ldots\ldots {}^mC_k\,{}^nC_0 = {}^{(m+n)}C_k,$
where m, n, k are positive integers, and ${}^pC_q = 0$ for $p < q$.

11. Prove that *(1989 - 5 Marks)*
$C_0 - 2^2C_1 + 3^2C_2 - \ldots\ldots\ldots\ldots + (-1)^n(n+1)^2C_n = 0,$
$n > 2$, where $C_r = {}^nC_r$.

12. Prove that $\frac{n^7}{7} + \frac{n^5}{5} + \frac{2n^3}{3} - \frac{n}{105}$ is an integer for every positive integer n. *(1990 - 2 Marks)*

13. Using induction or otherwise, prove that for any non-negative integers m, n, r and k, *(1991 - 4 Marks)*
$$\sum_{m=0}^{k}(n-m)\frac{(r+m)!}{m!} = \frac{(r+k+1)!}{k!}\left[\frac{n}{r+1} - \frac{k}{r+2}\right]$$

14. If $\sum_{r=0}^{2n} a_r(x-2)^r = \sum_{r=0}^{2n} b_r(x-3)^r$ and $a_k = 1$ for all $k \ge n$, then show that $b_n = {}^{2n+1}C_{n+1}$ *(1992 - 6 Marks)*

15. Let $p \ge 3$ be an integer and α, β be the roots of $x^2 - (p+1)x + 1 = 0$ using mathematical induction show that $\alpha^n + \beta^n$.
(i) is an integer and (ii) is not divisible by p *(1992 - 6 Marks)*

16. Using mathematical induction, prove that
$\tan^{-1}(1/3) + \tan^{-1}(1/7) + \ldots\ldots \tan^{-1}\{1/(n^2+n+1)\}$
$= \tan^{-1}\{n/(n+2)\}$ *(1993 - 5 Marks)*

17. Prove that $\sum_{r=1}^{k}(-3)^{r-1}\,{}^{3n}C_{2r-1} = 0$, where $k = (3n)/2$ and n is an even positive integer. *(1993 - 5 Marks)*

18. If x is not an integral multiple of 2π use mathematical induction to prove that : *(1994 - 4 Marks)*
$$\cos x + \cos 2x + \ldots\ldots\ldots + \cos nx = \cos\frac{n+1}{2}x \sin\frac{nx}{2}\operatorname{cosec}\frac{x}{2}$$

19. Let n be a positive integer and *(1994 - 5 Marks)*
$(1+x+x^2)^n = a_0 + a_1x + \ldots\ldots\ldots\ldots + a_{2n}x^{2n}$
Show that $a_0^2 - a_1^2 + a_2^2 \ldots\ldots\ldots\ldots + a_{2n}^2 = a_n$

20. Using mathematical induction prove that for every integer $n \ge 1$, $(3^{2n} - 1)$ is divisible by 2^{n+2} but not by 2^{n+3}. *(1996 - 3 Marks)*

21. Let $0 < A_i < \pi$ for $i = 1, 2 \ldots\ldots, n$. Use mathematical induction to prove that
$$\sin A_1 + \sin A_2 \ldots + \sin A_n \le n \sin\left(\frac{A_1 + A_2 + \ldots\ldots + A_n}{n}\right)$$
where ≥ 1 is a natural number.
{You may use the fact that
$p\sin x + (1-p)\sin y \le \sin[px + (1-p)y]$,
where $0 \le p \le 1$ and $0 \le x, y \le \pi$.} *(1997 - 5 Marks)*

22. Let p be a prime and m a positive integer. By mathematical induction on m, or otherwise, prove that whenever r is an integer such that p does not divide r, p divides ${}^{mp}C_r$. *(1998 - 8 Marks)*
[Hint: You may use the fact that $(1+x)^{(m+1)p} = (1+x)^p(1+x)^{mp}$]

23. Let n be any positive integer. Prove that **(1999 - 10 Marks)**

$$\sum_{k=0}^{m}\frac{\binom{2n-k}{k}}{\binom{2n-k}{n}}\cdot\frac{(2n-4k+1)}{(2n-2k+1)}2^{n-2k}=\frac{\binom{n}{m}}{\binom{2n-2m}{n-m}}2^{n-2m}$$

for each non-be gatuve integer $m \le n$. $\left(\text{Here}\binom{p}{q}={}^pC_q\right)$.

24. For any positive integer m, n (with $n \ge m$), let $\binom{n}{m}={}^nC_m$.

Prove that $\binom{n}{m}+\binom{n-1}{m}+\binom{n-2}{m}+.....+\binom{m}{m}=\binom{n+1}{m+2}$.

Hence or otherwise, prove that

$\binom{n}{m}+2\binom{n-1}{m}+3\binom{n-2}{m}+.....+(n-m+1)\binom{m}{m}=\binom{n+2}{m+2}$. **(2000 - 6 Marks)**

25. For every positive integer n, prove that $\sqrt{(4n+1)}<\sqrt{n}+\sqrt{n+1}<\sqrt{4n+2}$. Hence or otherwise, prove that $[\sqrt{n}+\sqrt{(n+1)}]=[\sqrt{4n+1}]$, where $[x]$ denotes the greatest integer not exceeding x. **(2000 - 6 Marks)**

26. Let a, b, c be positive real numbers such that $b^2-4ac>0$ and let $\alpha_1=c$. Prove by induction that

$\alpha_{n+1}=\dfrac{a\alpha_n^2}{(b^2-2a(\alpha_1+\alpha_2+...+\alpha_n)}$ is well – defined and

$\alpha_{n+1}<\dfrac{\alpha_n}{2}$ for all $n=1, 2, \ldots$ (Here, 'well – defined' means that the denominator in the expression for α_{n+1} is not zero.) **(2001 - 5 Marks)**

27. Use mathematical induction to show that $(25)^{n+1}-24n+5735$ is divisible by $(24)^2$ for all $n=1, 2, \ldots\ldots$ **(2002 - 5 Marks)**

28. Prove that **(2003 - 2 Marks)**

$$2^k\binom{n}{0}\binom{n}{k}-2^{k-1}\binom{n}{1}\binom{n-1}{k-1}+2^{k-2}\binom{n-2}{k-2}-.....(-1)^k\binom{n}{k}\binom{n-k}{0}=\binom{n}{k}.$$

29. A coin has probability p of showing head when tossed. It is tossed n times. Let p_n denote the probability that no two (or more) consecutive heads occur. Prove that $p_1=1$, $p_2=1-p^2$ and $p_n=(1-p).\ p_{n-1}+p(1-p)\,p_{n-2}$ for all $n\ge 3$.

Prove by induction on n, that $p_n=A\alpha^n+B\beta^n$ for all $n\ge 1$, where α and β are the roots of quadratic equation

$x^2-(1-p)x-p(1-p)=0$ and $A=\dfrac{p^2+\beta-1}{\alpha\beta-\alpha^2}$, $B=\dfrac{p^2+\alpha-1}{\alpha\beta-\beta^2}$. **(2000 - 5 Marks)**

I Integer Value Correct Type

1. The coefficients of three consecutive terms of $(1+x)^{n+5}$ are in the ratio 5 : 10 : 14. Then n = **(JEE Adv. 2013)**

2. Let m be the smallest positive integer such that the coefficient of x^2 in the expansion of $(1+x)^2+(1+x)^3+...+(1+x)^{49}+(1+mx)^{50}$ is $(3n+1)\ {}^{51}C_3$ for some positive integer n. Then the value of n is **(JEE Adv. 2016)**

Section-B JEE Main / AIEEE

1. The coefficients of x^p and x^q in the expansion of $(1+x)^{p+q}$ are **[2002]**
(a) equal
(b) equal with opposite signs
(c) reciprocals of each other
(d) none of these

2. If the sum of the coefficients in the expansion of $(a+b)^n$ is 4096, then the greatest coeficient in the expansion is **[2002]**
(a) 1594 (b) 792 (c) 924 (d) 2924

3. The positive integer just greater than $(1+0.0001)^{10000}$ is **[2002]**
(a) 4 (b) 5 (c) 2 (d) 3

4. r and n are positive integers $r>1$, $n>2$ and coefficient of $(r+2)^{th}$ term and $3r^{th}$ term in the expansion of $(1+x)^{2n}$ are equal, then n equals **[2002]**
(a) $3r$ (b) $3r+1$ (c) $2r$ (d) $2r+1$

5. If $a_n=\sqrt{7+\sqrt{7+\sqrt{7+\ldots\ldots}}}$ haing n radical signs then by methods of mathematical induction which is true **[2002[**
(a) $a_n>7\ \forall\ n\ge 1$ (b) $a_n<7\ \forall\ n\ge 1$
(c) $a_n<4\ \forall\ n\ge 1$ (d) $a_n<3\ \forall\ n\ge 1$

6. If x is positive, the first negative term in the expansion of $(1+x)^{27/5}$ is **[2003]**
(a) 6th term (b) 7th term (c) 5th term (d) 8th term.

7. The number of integral terms in the expansion of $(\sqrt{3}+\sqrt[8]{5})^{256}$ is **[2003]**
(a) 35 (b) 32 (c) 33 (d) 34

8. Let $S(K)=1+3+5...+(2K-1)=3+K^2$. Then which of the following is true **[2004]**
(a) Principle of mathematical induction can be used to prove the formula
(b) $S(K)\Rightarrow S(K+1)$
(c) $S(K)\not\Rightarrow S(K+1)$
(d) $S(1)$ is correct

9. The coefficient of the middle term in the binomial expansion in powers of x of $(1+\alpha x)^4$ and of $(1-\alpha x)^6$ is the same if α equals **[2004]**
(a) $\frac{3}{5}$ (b) $\frac{10}{3}$ (c) $\frac{-3}{10}$ (d) $\frac{-5}{3}$

10. The coefficient of x^n in expansion of $(1+x)(1-x)^n$ is
(a) $(-1)^{n-1}n$ (b) $(-1)^n(1-n)$ [2004]
(c) $(-1)^{n-1}(n-1)^2$ (d) $(n-1)$

11. The value of ${}^{50}C_4+\sum_{r=1}^{6}{}^{56-r}C_3$ is [2005[
(a) ${}^{55}C_4$ (b) ${}^{55}C_3$ (c) ${}^{56}C_3$ (d) ${}^{56}C_4$

12. If $A=\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}$ and $I=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$, then which one of the following holds for all $n\geq 1$, by the principle of mathematical induction [2005]
(a) $A^n=nA-(n-1)I$ (b) $A^n=2^{n-1}A-(n-1)I$
(c) $A^n=nA+(n-1)I$ (d) $A^n=2^{n-1}A+(n-1)I$

13. If the coefficient of x^7 in $\left[ax^2+\left(\frac{1}{bx}\right)\right]^{11}$ equals the coefficient of x^{-7} in $\left[ax-\left(\frac{1}{bx^2}\right)\right]^{11}$, then a and b satisfy the relation [2005]
(a) $a-b=1$ (b) $a+b=1$
(c) $\frac{a}{b}=1$ (d) $ab=1$

14. If x is so small that $x^3$and higher powers of x may be neglected, then $\dfrac{(1+x)^{\frac{3}{2}}-\left(1+\frac{1}{2}x\right)^3}{(1-x)^{\frac{1}{2}}}$ may be approximated as
(a) $1-\frac{3}{8}x^2$ (b) $3x+\frac{3}{8}x^2$ [2005]
(c) $-\frac{3}{8}x^2$ (d) $\frac{x}{2}-\frac{3}{8}x^2$

15. If the expansion in powers of x of the function $\dfrac{1}{(1-ax)(1-bx)}$ is $a_0+a_1x+a_2x^2+a_3x^3$...... then a_n is
(a) $\dfrac{b^n-a^n}{b-a}$ (b) $\dfrac{a^n-b^n}{b-a}$ [2006]
(c) $\dfrac{a^{n+1}-b^{n+1}}{b-a}$ (d) $\dfrac{b^{n+1}-a^{n+1}}{b-a}$

16. For natural numbers m, n if $(1-y)^m(1+y)^n$ $=1+a_1y+a_2y^2+$....... and $a_1=a_2=10$, then (m,n) is
(a) $(20,45)$ (b) $(35,20)$ [2006]
(c) $(45,35)$ (d) $(35,45)$

17. In the binomial expansion of $(a-b)^n$, $n\geq 5$, the sum of 5th and 6th terms is zero, then a/b equals [2007[
(a) $\dfrac{n-5}{6}$ (b) $\dfrac{n-4}{5}$ (c) $\dfrac{5}{n-4}$ (d) $\dfrac{6}{n-5}$.

18. The sum of the series [2007]
${}^{20}C_0-{}^{20}C_1+{}^{20}C_2-{}^{20}C_3+$..... –.....+ ${}^{20}C_{10}$ is
(a) 0 (b) ${}^{20}C_{10}$ (c) $-{}^{20}C_{10}$ (d) $\frac{1}{2}\,{}^{20}C_{10}$

19. **Statement -1 :** $\sum_{r=0}^{n}(r+1)\,{}^nC_r=(n+2)2^{n-1}$. [2008]
Statement-2 : $\sum_{r=0}^{n}(r+1)\,{}^nC_rx^r=(1+x)^n+nx(1+x)^{n-1}$.
(a) Statement -1 is false, Statement-2 is true
(b) Statement -1 is true, Statement-2 is true; Statement -2 is a correct explanation for Statement-1
(c) Statement -1 is true, Statement-2 is true; Statement -2 is not a correct explanation for Statement-1
(d) Statement -1 is true, Statement-2 is false

20. The remainder left out when $8^{2n}-(62)^{2n+1}$ is divided by 9 is : [2009]
(a) 2 (b) 7 (c) 8 (d) 0

21. Let $S_1=\sum_{j=1}^{10}j(j-1)^{10}C_J$, $S_2=\sum_{j=1}^{10}j^{10}C_j$ and $S_3=\sum_{j=1}^{10}j^2\,{}^{10}C_j$.
Statement-1 : $S_3=55\times 2^9$.
Statement-2: $S_1=90\times 2^8$ and $S_2=10\times 2^8$. [2010]
(a) Statement -1 is true, Statement -2 is true ; Statement -2 is **not** a correct explanation for Statement -1.
(b) Statement -1 is true, Statement -2 is false.
(c) Statement -1 is false, Statement -2 is true .
(d) Statement - 1 is true, Statement 2 is true ; Statement -2 is a correct explanation for Statement -1.

22. The coefficient of x^7 in the expansion of $(1-x-x^2+x^3)^6$ is [2011]
(a) –132 (b) –144 (c) 132 (d) 144

23. If n is a positive integer , then $\left(\sqrt{3}+1\right)^{2n}-\left(\sqrt{3}-1\right)^{2n}$ is :
(a) an irrational number [2012]
(b) an odd positive integer
(c) an even positive integer
(d) a rational number other than positive integers

24. The term independent of x in expansion of $\left(\dfrac{x+1}{x^{2/3}-x^{1/3}+1}-\dfrac{x-1}{x-x^{1/2}}\right)^{10}$ is [JEE M 2013]
(a) 4 (b) 120 (c) 210 (d) 310

25. If the coefficents of x^3 and x^4 in the expansion of $(1+ax+bx^2)(1-2x)^{18}$ in powers of x are both zero, then (a,b) is equal to: [JEE M 2014]
(a) $\left(14,\frac{272}{3}\right)$ (b) $\left(16,\frac{272}{3}\right)$ (c) $\left(16,\frac{251}{3}\right)$ (d) $\left(14,\frac{251}{3}\right)$

26. The sum of coefficients of integral power of x in the binomial expansion $\left(1-2\sqrt{x}\right)^{50}$ is : [JEE M 2015]
(a) $\frac{1}{2}(3^{50}-1)$ (b) $\frac{1}{2}(2^{50}+1)$
(c) $\frac{1}{2}(3^{50}+1)$ (d) $\frac{1}{2}(3^{50})$

27. If the number of terms in the expansion of $\left(1-\frac{2}{x}+\frac{4}{x^2}\right)^n$, $x\neq 0$, is 28, then the sum of the coefficients of all the terms in this expansion, is : [JEE M 2016]
(a) 243 (b) 729 (c) 64 (d) 2187

5 Mathematical Induction and Binomial Theorem

Section-A : JEE Advanced/ IIT-JEE

A	1.	$(101)^{50}$	2.	-1	3.	$a=2, n=4$	4.	7	5.	41		
C	1.	(a)	2.	(a)	3.	(c)	4.	(c)	5.	(d)	6.	(b)
	7.	(c)	8.	(d)	9.	(d)	10.	(a)	11.	(d)	12.	(c)
D	1.	(c)	2.	(c)								
I	1.	6	2.	5								

Section-B : JEE Main/ AIEEE

1.	(a)	2.	(c)	3.	(d)	4.	(c)	5.	(b)	6.	(d)
7.	(c)	8.	(b)	9.	(c)	10.	(b)	11.	(d)	12.	(a)
13.	(d)	14.	(c)	15.	(d)	16.	(d)	17.	(b)	18.	(d)
19.	(b)	20.	(a)	21.	(b)	22.	(b)	23.	(a)	24.	(c)
25.	(b)	26.	(c)	27.	(b)						

Section-A JEE Advanced/ IIT-JEE

A. Fill in the Blanks

1. Consider $(101)^{50} - \{(99)^{50} + (100)^{50}\}$
$= (100+1)^{50} - (100-1)^{50} - (100)^{50}$
$= (100)^{50}[(1+0.01)^{50} - (1-0.01)^{50} - 1]$
$= (100)^{50}[2\,({}^{50}C_1(0.01) + {}^{50}C_3(0.01)^3 + ...) - 1]$
$= (100)^{50}[2\,({}^{50}C_3(0.01)^3 + ...)] > 0$
$\therefore\ (101)^{50} > (99)^{50} + (100)^{50}\quad \therefore\ (101)^{50}$ is greater.

2. If we put $x = 1$ in the expansion of $(1+x-3x^2)^{2163} = A_0 + A_1x + A_2x^2 + ...$ we will get the sum of coefficients of given polynomial, which clearly comes to be -1.

3. $(1+ax)^n = 1 + 8x + 24x^2 + ...$

$$\Rightarrow (1+ax)^n = 1 + nxa + \frac{n(n-1)}{2!}.a^2x^2 + ...$$
$$= 1 + 8x + 24x^2 + ...$$

Comparing like powers of x we get
$nax = 8x \;\Rightarrow\; na = 8$ ....(1)

$$\frac{n(n-1)a^2}{2} = 24 \;\Rightarrow\; n(n-1)a^2 = 48 \qquad ....(2)$$

Solving (1) and (2), $n = 4,\ a = 2$

4. We know that for a +ve integer n
$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + ...+ {}^nC_nx^n$
ATQ coefficients of 2^{nd}, 3^{rd}, and 4^{th} terms are in A.P.
i.e. ${}^nC_1, {}^nC_2, {}^nC_3$ are in A.P.
$\Rightarrow 2.{}^nC_2 = {}^nC_1 + {}^nC_3$

$$\Rightarrow 2 \times \frac{n(n-1)}{2} = n + \frac{n(n-1)\,(n-2)}{3!}$$

$$\Rightarrow n - 1 = 1 + \frac{n^2 - 3n + 2}{6} \Rightarrow n^2 - 9n + 14 = 0$$

$\Rightarrow (n-7)(n-2) = 0 \Rightarrow n = 7$ or 2
But for the existance of 4^{th} term, $n = 7$.

5. Let T_{r+1} be the general term in the expansion of $(\sqrt{2} + 3^{1/5})^{10}$

$$\therefore\ T_{r+1} = {}^{10}C_r(\sqrt{2})^{10-r}.(3^{1/5})^r.(0 \le r \le 10)$$
$$= \frac{10!}{r!(10-r)!}.2^{5-r/2}.3^{r/5}$$

Let T_{r+1} will be rational if $2^{5-r/2}$ and $3^{r/5}$ are rational numbers.

$\Rightarrow 5 - \dfrac{r}{2}$ and $\dfrac{r}{5}$ are integers.
$\Rightarrow$ r = 0 and r = 10 $\Rightarrow T_1$ and T_{11} are rational terms.
$\Rightarrow$ Sum of T_1 and $T_{11} = {}^{10}C_0 2^{5-0}.3^0 + {}^{10}C_{10}2^{5-5}.3^2$
$= 1.32.1 + 1.1.9 = 32 + 9 = 41$

C. MCQs with ONE Correct Answer

1. **(a)** Given that r and n are +ve integers such that $r > 1, n > 2$
Also in the expansion of $(1+x)^{2n}$
coeff. of $(3r)^{th}$ term = coeff. of $(r+2)^{th}$ term
$\Rightarrow {}^{2n}C_{3r-1} = {}^{2n}C_{r+1}$
$\Rightarrow 3r - 1 = r + 1$ or $3r - 1 + r + 1 = 2n$
$\Rightarrow r = 1$ or $2r = n$
But $r > 1 \quad \therefore\ n = 2r$

2. **(a)** General term in the expansion $\left(\dfrac{x}{2} - \dfrac{3}{x^2}\right)^{10}$ is

$$T_{r+1} = {}^{10}C_r\left(\frac{x}{2}\right)^{10-r}\left(\frac{-3}{x^2}\right)^r = {}^{10}C_r x^{10-3r}\frac{(-1)^r 3^r}{2^{10-r}}$$

For coeff of x^4, we should have

$10 - 3r = 4 \Rightarrow r = 2$

$\therefore$ Coeff of $x^4 = {}^{10}C_2 \dfrac{(-1)^2 3^2}{2^8} = \dfrac{405}{256}$

3. (c) The given expression is

$(x + \sqrt{x^3 - 1})^5 + (x - \sqrt{x^3 - 1})^5$

We know by binomial theorem, that

$(x + a)^n + (x - a)^n = 2\,[{}^nC_0 x^n + {}^nC_2 x^{n-2} a^2 + {}^nC_4 x^{n-4} a^4 + \ldots\ldots]$

$\therefore$ The given expression is equal to

$2\,[{}^5C_0 x^5 + {}^5C_2 x^3 (x^3 - 1) + {}^5C_4 x\,(x^3 - 1)^2]$

Max. power of x involved here is 7, also only +ve integral powers of x are involved, therefore given expression is a polynomial of degree 7.

4. (c) We have $(1 + x)^m\,(1 - x)^n$

$\left[1 + mx + \dfrac{m(m-1)}{2!}x^2 + \ldots\right]\left[1 - nx + \dfrac{n(n-1)}{2!}x^2 - \ldots\right]$

$= 1 + (m - n)\,x + \left[\dfrac{m(m-1)}{2} + \dfrac{n(n-1)}{2} - mn\right]x^2 + \ldots$

Given, $m - n = 3$ ....(1)

and $\dfrac{1}{2}m(m-1) + \dfrac{1}{2}n(n-1) - mn = -6$

$\Rightarrow m^2 + n^2 - 2mn - (m + n) = -12$

$\Rightarrow (m - n)^2 - (m + n) = -12$

$\Rightarrow m + n = 9 + 12 = 21$ ....(2)

From (1) and (2), we get $m = 12$

5. (d) $\binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2}$

$= \left[\binom{n}{r} + \binom{n}{r-1}\right] + \left[\binom{n}{r-1} + \binom{n}{r-2}\right]$

NOTE THIS STEP : $\binom{n+1}{r} + \binom{n+1}{r-1} = \binom{n+2}{r}$

$[\because\ {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r]$

6. (b) $(a - b)^n, n \geq 5$

In binomial expansion of above $T_5 + T_6 = 0$

$\Rightarrow {}^nC_4\, a^{n-4} b^4 + {}^nC_5\, a^{n-5} b^5 = 0$

$\Rightarrow \dfrac{{}^nC_4}{{}^nC_5}\cdot\dfrac{a}{b} = 1 \Rightarrow \dfrac{4+1}{n-4}\cdot\dfrac{a}{b} = 1 \Rightarrow \dfrac{a}{b} = \dfrac{n-4}{5}$

7. (c) $\sum_{i=0}^{m} {}^{10}C_i\,{}^{20}C_{m-i} = {}^{10}C_0\,{}^{20}C_m + {}^{10}C_1\,{}^{20}C_{m-1} + {}^{10}C_2\,{}^{20}C_{m-2} + \ldots\ldots + {}^{10}C_m\,{}^{20}C_0$

= Coeff of x^m in the expansion of product $(1 + x)^{10}(1 + x)^{20}$

= Coeff of x^m in the expansion of $(1 + x)^{30}$

$= {}^{30}C_m$

To get max. value of given sum, ${}^{30}C_m$ should be max. which is so when $m = 30/2 = 15$.

$\left[\text{Using the fact that max } ({}^nC_r) = \begin{cases} {}^nC_{n/2} \text{ if } n \text{ is even} \\ {}^nC_{\frac{n+1}{2}} \text{ if } n \text{ is odd} \end{cases}\right]$

8. (d) $(1 + t^2)^{12}\,(1 + t^{12})\,(1 + t^{24})$

$= (1 + t^{12} + t^{24} + t^{36})\,(1 + t^2)^{12}$

$\therefore$ Coeff. of $t^{24} = 1\times$ Coeff. of t^{24} in $(1 + t^2)^{12} + 1\times$ Coeff. of t^{12} in $(1 + t^2)^{12} + 1\times$ constant term in $(1 + t^2)^{12}$

$= {}^{12}C_{12} + {}^{12}C_6 + {}^{12}C_0 = 1 + {}^{12}C_6 + 1 = {}^{12}C_6 + 2$

9. (d) ${}^{n-1}C_r = {}^nC_{r+1}(k^2 - 3) \Rightarrow k^2 - 3 = \dfrac{{}^{n-1}C_r}{{}^nC_{r+1}} = \dfrac{r+1}{n}$

Since $0 \leq r \leq n - 1$

$\Rightarrow 1 \leq r + 1 \leq n \Rightarrow \dfrac{1}{n} \leq \dfrac{r+1}{n} \leq 1 \Rightarrow \dfrac{1}{n} \leq k^2 - 3 \leq 1$

$\Rightarrow 3 + \dfrac{1}{n} \leq k^2 \leq 4 \Rightarrow \sqrt{3 + \dfrac{1}{n}} \leq k \leq 2$

as $n \to \infty \Rightarrow \sqrt{3} < k \leq 2 \Rightarrow k \in (\sqrt{3}, 2]$

10. (a) To find

${}^{30}C_0\,{}^{30}C_{10} - {}^{30}C_1\,{}^{30}C_{11} + {}^{30}C_2\,{}^{30}C_{12} - \ldots + {}^{30}C_{20}\,{}^{30}C_{30}$

We know that

$(1 + x)^{30} = {}^{30}C_0 + {}^{30}C_1 x + {}^{30}C_2 x^2 + \ldots + {}^{30}C_{20} x^{20} + \ldots {}^{30}C_{30} x^{30}$ ....(1)

$(x - 1)^{30} = {}^{30}C_0 x^{30} - {}^{30}C_1 x^{29} + \ldots + {}^{30}C_{10} x^{20} - {}^{30}C_{11} x^{19} + {}^{30}C_{12} x^{18} + \ldots {}^{30}C_{30} x^0$ ....(2)

Multiplying eq^n (1) and (2), we get

$(x^2 - 1)^{30} = (\ ) \times (\ )$

Equating the coefficients of x^{20} on both sides, we get

${}^{30}C_{10} = {}^{30}C_0\,{}^{30}C_{10} - {}^{30}C_1\,{}^{30}C_{11} + {}^{30}C_2\,{}^{30}C_{12} - \ldots + {}^{30}C_{20}\,{}^{30}C_{30}$

$\therefore$ Req. value is ${}^{30}C_{10}$

11. (d) Clearly $A_r = {}^{10}C_r, B_r = {}^{20}C_r, C_r = {}^{30}C_r$

Now $\sum_{r=1}^{10} {}^{10}C_r\left({}^{20}C_{10}\,{}^{20}C_r - {}^{30}C_{10}\,{}^{10}C_r\right)$

$= {}^{20}C_{10} \sum_{r=1}^{10} {}^{10}C_r\,{}^{20}C_r - {}^{30}C_{10} \sum_{r=1}^{10} {}^{10}C_r \times {}^{10}C_r$

$= {}^{20}C_{10}\left({}^{10}C_1\,{}^{20}C_1 + {}^{10}C_2\,{}^{20}C_2 + \ldots + {}^{10}C_{10}\,{}^{20}C_{10}\right)$

$- {}^{30}C_{10}\left({}^{10}C_1 \times {}^{10}C_1 + {}^{10}C_2 \times {}^{10}C_2 + \ldots + {}^{10}C_{10}\,{}^{10}C_{10}\right)$ ....(1)

Now expanding $(1 + x)^{10}$ and $(1 + x)^{20}$ by binomial theorem and comparing the coefficients of x^{20} in their product, on both sides, we get

${}^{10}C_0\,{}^{20}C_0 + {}^{10}C_1\,{}^{20}C_1 + {}^{10}C_2\,{}^{20}C_2 + \ldots + {}^{10}C_{10}\,{}^{20}C_{10}$

$=$ coeff of x^{20} in $(1 + x)^{30} = {}^{30}C_{20} = {}^{30}C_{10}$

$\therefore {}^{10}C_1\,{}^{20}C_1 + {}^{10}C_2\,{}^{20}C_2 + \ldots + {}^{10}C_{10}\,{}^{20}C_{10} = {}^{30}C_{10} - 1$

Again expending $(1 + x)^{10}$ and $(x + 1)^{10}$ by binomial theorem and comparing the coefficients of x^{10} in their product on both sides, we get

$\therefore\ \left({}^{10}C_0\right)^2\left({}^{10}C_1\right)^2+\left({}^{10}C_2\right)^2+...+\left({}^{10}C_{10}\right)^2=$

coeff of x^{10} in $(1+x)^{20}={}^{20}C_{10}$

$\therefore\ \left({}^{10}C_1\right)^2+\left({}^{10}C_2\right)^2+...+\left({}^{10}C_{10}\right)^2={}^{20}C_{10}-1$

Substituting these values in equation (1), we get

$={}^{20}C_{10}\left({}^{30}C_{10}-1\right)-{}^{30}C_{10}\left({}^{20}C_{10}-1\right)$

$={}^{30}C_{10}-{}^{20}C_{10}=C_{10}-B_{10}$

12. (c) Coeff. of x^{11} in exp. of $\left(1+x^2\right)^4\left(1+x^3\right)^7\left(1+x^4\right)^{12}$

$=$ (Coeff. of x^a) $\times$ (Coeff. of x^b) $\times$ (Coeff. of x^c)

Such that $a+b+c=11$

Here $a=2m,\ b=3n,\ c=4p$

$\therefore\ 2m+3n+4p=11$

Case I : $m=0, n=1, p=2$

Case II : $m=1, n=3, p=0$

Case III : $m=2, n=1, p=1$

Case IV : $m=4, n=1, p=0$

$\therefore$ Required coeff.

$={}^4C_0\times{}^7C_1\times{}^{12}C_2+{}^4C_1\times{}^7C_3\times{}^{12}C_0$

$+{}^4C_2\times{}^7C_1\times{}^{12}C_1+{}^4C_4\times{}^7C_1\times{}^{12}C_0$

$=462+140+504+7=1113$

D. MCQs with ONE or MORE THAN ONE Correct

1. (c) $\because$ n is even, let $n=2m$ then

$$\text{LHL}=S=\frac{2.m!m!}{(2m)!}[C_0^2-2C_1^2+3C_2^2.....$$

$$+(-1)^{2m}(2m+1)C_{2m}^2....]\qquad ....(1)$$

$$=\frac{2.m!.m!}{(2m)!}C_{2m}^2-2C_{2m-1}^2+3C_{2m-2}^2-.....$$

$$+(-1)^{2m}(2m+1)C_0^2\qquad [\text{Using } C_r=C_{n-r}]$$

$$\Rightarrow\ S=\frac{2.m!m!}{(2m)!}[(2m+1)C_0^2-2mC_1^2$$

$$+(2m-1)C_2^2......-2C_{2m-1}^2+C_{2m}^2]\qquad ....(2)$$

Adding (1) and (2):

$$2S=2\frac{m!m!}{(2m)!}[2m+2][C_0^2-C_1^2+C_2^2+.....+C_{2m}^2]$$

Now keeping in mind that if n is even, then

$$C_0^2-C_1^2+C_2^2-......+C_n^2=(-1)^{n/2}\ {}^nC_{n/2}$$

$\therefore$ we get

$$S=\frac{m!m!}{(2m)!}(2m+2)[(-1)^m\ {}^{2m}C_m]=\left(2\frac{n}{2}+2\right)(-1)^{n/2}$$

$$=(-1)^{n/2}(n+2)$$

2. (c) Let $b=\sum_{r=0}^{n}\frac{r}{{}^nC_r}=\sum_{r=0}^{n}\frac{n-(n-r)}{{}^nC_r}$

$$=na_n-\sum_{r=0}^{n}\frac{n-r}{{}^nC_{n-r}}\qquad [\because\ {}^nC_r={}^nC_{n-r}]$$

$$=na_n-b$$

$$\Rightarrow\ 2b=na_n\ \Rightarrow\ b=\frac{n}{2}a_n$$

E. Subjective Problems

1. Given that

$C_1+2C_2x+3C_3x^2+....+2nC_{2n}x^{2n-1}=2n(1+x)^{2n-1}$ ....(1)

where $C_r=\dfrac{2n!}{r!(2n-r)!}$

Integrating both sides with respect to x, under the limits 0 to x, we get

$[C_1x+C_2x^2+C_3x^3+....+C_{2n}x^{2n}]_0^x=[(1+x)^{2n}]_0^x$

$\Rightarrow\ C_1x+C_2x^2+C_3x^3+....+C_{2n}x^{2n}=(1+x)^{2n}-1$

$\Rightarrow\ C_0+C_1x+C_2x^2+C_3x^3+....+C_{2n}x^{2n}=(1+x)^{2n}$ ....(2)

Changing x by $-\dfrac{1}{x}$, we get

$$\Rightarrow\ C_0-\frac{C_1}{x}+\frac{C_2}{x^2}-\frac{C_3}{x^3}+....+(-1)^{2n}\frac{C_{2n}}{x^{2n}}=\left(1-\frac{1}{x}\right)^{2n}$$

$\Rightarrow\ C_0x^{2n}-C_1x^{2n-1}+C_2x^{2n-2}-C_3x^{2n-3}$

$+....+C_{2n}=(x-1)^{2n}$ ....(3)

Multiplying eqn. (1) and (3) and equating the coefficients of x^{2n-1} on both sides, we get

$-C_1^2+2C_2^2-3C_3^2+....+2nC_{2n}^2$

$=$ coeff.of x^{2n-1} in $2n(x-1)(x^2-1)^{2n-1}$

$=2n$ [coeff. of x^{2n-2} in $(x^2-1)^{2n-1}$

$-$ coeff. of x^{2n-1} in $(x^2-1)^{2n-1}$]

$=2n[{}^{2n-1}C_{n-1}(-1)^{n-1}-0]$

$=(-1)^{n-1}.2n\ {}^{2n-1}C_{n-1}$

$\Rightarrow\ C_1^2-2C_2^2+3C_3^2+....+2nC_{2n}^2$

$$=(-1)^n.2n\ {}^{2n-1}C_{n-1}=(-1)^n n.\left(\frac{2n}{n}.{}^{2n-1}C_{n-1}\right)$$

$$=(-1)^n n.{}^{2n}C_n=(-1)^n n.C_n.\qquad (\because\ {}^{2n}C_n=C_n)$$

Hence Proved.

2. $P(n)$: $7^{2n}+2^{3n-3},3^{n-1}$ is divisible by 25 $\forall n\in N$.

Let us prove it by Mathematical Induction :

$P(1)$: $7^2+2^0.3^0=49+1=50$ which is divisible by 25.

$\therefore$ $P(1)$ is true.

Let $P(k)$ be true that is $7^{2k}+2^{3k-3}, 3^{k-1}$ is divisible by 25.

$\Rightarrow\ 7^{2k}+2^{3k-3}.3^{k-1}=25m$ where $m\in Z$.

$\Rightarrow\ 2^{3k-3}.3^{k-1}=25m-7^{2k}$ ....(1)

Consider $P(k+1)$:

$7^{2(k+1)}+2^{3(k+1)-3}.3^{k+1-1}=7^{2k}.7^2+2^{3k}.3^k$

$=49.7^{2k}+2^3.3.2^{3k-3}.3^{k-1}=49.7^{2k}+24(25m-7^{2k})$

(Using IH eq. (1))

$=49.7^{2k}+24\times25m-24\times7^{2k}$

$=25.7^{2k}+24\times25m=25(7^{2k}+24m)$

$=25\times$ some integral value which is divisible by 25.

$\therefore$ $P(k+1)$ is also true.

Hence by the principle of mathematical induction

$P(n)$ is true $\forall\ n \in Z$.

3. $S = \sum \sum C_i C_j$

$0 \le i < j \le n$

NOTE THIS STEP

$\Rightarrow S = C_0(C_1 + C_2 + C_3 + ... + C_n) + C_1(C_2 + C_3 + ... + C_n) + C_2(C_3 + C_4 + C_5 + ... C_n) + ... C_{n-1}(C_n)$

$\Rightarrow S = C_0(2^n - C_0) + C_1(2^n - C_0 - C_1) + C_2(2^n - C_0 - C_1 - C_2) + ... + C_{n-1}(2^n - C_0 - C_1 .... C_{n-1}) + C_n(2^n - C_0 - C_1 .... C_n)$

$\Rightarrow S = 2^n(C_0 + C_1 + C_2 + ... + C_{n-1} + C_n) - (C_0^2 + C_1^2 + C_2^2 + ... + C_n^2) - S$

$$\Rightarrow 2S = 2^n . 2^n - \frac{2n!}{(n!)^2} = 2^{2n} - \frac{2n!}{(n!)^2}$$

$$\Rightarrow S = 2^{2n-1} - \frac{2n!}{2(n!)^2}$$

4. $P(n) : n(n^2 - 1)$ is divisible by 24 for n odd +ve integer.

For $n = 2m - 1$, it can be restated as

$P(m) : (2m-1)(4m^2 - 4m) = 4m(m-1)(2m-1)$

is divisible by 24 $\forall\ m \in N$

$\Rightarrow P(m) : m(m-1)(2m-1)$ is divisible by 6 $\forall\ m \in N$.

Here $P(1) = 0$, divisible by 6.

$\therefore$ $P(1)$ is true.

Let it be true for $m = k$, i.e.,

$k(k-1)(2k-1) = 6p$

$\Rightarrow 2k^3 - 3k^2 + k = 6p$ ....(1)

Consider $P(k+1) : k(k+1)(2k+1) = 2k^3 + 3k^2 + k$

$= 6p + 3k^2 + 3k^2$ (Using (1)

$= 6(p + k^2) \Rightarrow$ divisible by 6

$\therefore$ $P(k+1)$ is also true.

Hence $P(m)$ is true $\forall\ m \in N$.

5. $P(n) : P^{n+1} + (p+1)^{2n-1}$ is divisible by $p^2 + p + 1$

For $n = 1$, $P(1) : p^2 + p + 1$ which is divisible by $p^2 + p + 1$.

$\therefore$ $P(1)$ is true.

Let $P(k)$ be true, i.e.,

$p^{k+1} + (p+1)^{2k-1}$ is divisible by $p^2 + p + 1$

$\Rightarrow p^{k+1} + (p+1)^{2k-1} = (p^2 + p + 1)m$ ....(1)

Consider $P(k+1) : p^{k+2} + (p+1)^{2k+1}$

$= p . p^{k+1} + (p+1)^{2k-1} . (p+1)^2$

$= p[m(p^2 + p + 1) - (p+1)^{2k-1}] + (p+1)^{2k-1}(p+1)^2$

$= p(p^2 + p + 1)m - p(p+1)^{2k-1} + (p+1)^{2k-1}(p^2 + 2p + 1)$

$= p(p^2 + p + 1)m + (p+1)^{2k-1}(p^2 + p + 1)$

$= (p^2 + p + 1)[mp + (p+1)^{2k-1}]$

$= (p^2 + p + 1)$ some integral value

$\therefore$ divisible by $p^2 + p + 1$ $\therefore$ $P(k+1)$ is also true.

Hence by principle of mathematical induction $P(n)$ is true $\forall\ n \in N$.

6. We have $s_n = \dfrac{1 - q^{n+1}}{1 - q}$ ....(1)

and $$S_n = \frac{1 - \left(\frac{q+1}{2}\right)^{n+1}}{1 - \left(\frac{q+1}{2}\right)} = \frac{2^{n+1} - (q+1)^{n+1}}{2^n(1-q)} \quad ....(2)$$

Now, ${}^{n+1}C_1 + {}^{n+1}C_2 s_1 + {}^{n+1}C_3 s_2 + ... + {}^{n+1}C_{n+1} s_n$

$$= \frac{1}{1-q}[{}^{n+1}C_1(1-q) + {}^{n+1}C_2(1-q^2) + {}^{n+1}C_3(1-q^3) + ... + ... + {}^{n+1}C_n(1-q^{n+1})] \quad \text{Using (1)}$$

$$= \frac{1}{1-q}\left[\left({}^{n+1}C_1 + {}^{n+1}C_2 + ... + {}^{n+1}C_{n+1}\right) - \left({}^{n+1}C_1 q + {}^{n+1}C_2 q^2 + ... + {}^{n+1}C_{n+1} q^{n+1}\right)\right]$$

$$= \frac{1}{1-q}\left[2^{n+1} - 1) - \{(1+q)^{n+1} - 1\}\right]$$

$$= \frac{2^{n+1} - (1+q)^{n+1}}{(1-q)} = 2^n S_n \quad \text{[Using eq. (2)]}$$

7. Let $A_n = 2.7^n + 3.5^n - 5$

Then $A_1 = 2.7 + 3.5 - 5 = 14 + 15 - 5 = 24$.

Hence A_1 is divisible by 24.

Now assume that A_m is divisible by 24 so that we may write

$A_m = 2.7^m + 3.5^m - 5 = 24k,\ k \in N$ ....(1)

Then $A_{m+1} - A_m = 2(7^{m+1} - 7^m) + 3(5^{m+1} - 5^m) - 5 + 5$

$= 2.7^m(7-1) + 3.5^m(5-1) = 12.(7^m + 5^m)$

Since 7^m and 5^m are odd integers $\forall\ m \in N$, their sum must be an even integer, say $7^m + 5^m = 2p, p \in N$.

Hence $A_{m+1} - A_m = 12.2p = 24p$

or $A_{m+1} = A_m + 24p = 24k + 24p$ [by (1)]

Hence A_{m+1} is divisible by 24.

It follows by mathematical induction that A_n is divisible by 24 for all $n \in N$.

8. Let $P(n) : \dfrac{(2n)!}{2^{2n}(n!)^2} \le \dfrac{1}{(3n+1)^{1/2}}$

For $n = 1$, $P(1) : \dfrac{2!}{2^2(1!)^2} \le \dfrac{1}{(3+1)^{1/2}} \Rightarrow \dfrac{1}{4} \le \dfrac{1}{2}$

$\Rightarrow \dfrac{1}{2} \le \dfrac{1}{2}$ which is true for $n = 1$

Assume that P(k) is true, then

$$P(k) : \frac{(2k)!}{2^{2k}(k!)^2} \le \frac{1}{(3k+1)^{1/2}} \quad ....(1)$$

For $n = k + 1$,

$$\frac{[2(k+1)]!}{2^{2(k+1)}[(k+1)!]^2} = \frac{(2k+2)!}{2^{2k+2}[(k+1)!]^2}$$

$$= \frac{(2k+2)(2k+1)(2k)!}{4.2^{2k}(k+1)^2(k!)^2}$$

$$\le \frac{(2k+2)(2k+1)}{4(k+1)^2} . \frac{1}{(3k+1)^{1/2}}$$

[Using Induction hypothesis (1)]

$$= \frac{(2k+1)}{2.(k+1)(3k+1)^{1/2}}$$

Thus, $$\frac{[2(k+1)]!}{2^{2(k+1)}[(k+1)!]^2} \le \frac{(2k+1)}{2(k+1)(3k+1)^{1/2}} \quad ....(2)$$

In order to prove $P(k+1)$, it is sufficient to prove that

$$\frac{(2k+1)}{2(k+1)\,(3k+1)^{1/2}} \le \frac{1}{(3k+4)^{1/2}} \quad ....(3)$$

Squaring eq. (3), we get

$$\frac{(2k+1)^2}{4(k+1)^2(3k+1)} \le \frac{1}{3k+4}$$

$\Rightarrow\ (2k+1)^2\,(3k+4) - 4\,(k+1)^2(3k+1) \le 0$

$\Rightarrow\ (4k^2+4k+1)\,(3k+4) - 4(k^2+2k+1)(3k+1) \le 0$

$\Rightarrow (12k^3+28k^2+19k+4) - (12k^3+28k^2+20k+4) \le 0$

$\Rightarrow\ -k\ \le 0$

which is true.

Hence from (2) and (3), we get

$$\frac{(2k+2)!}{2^{2k+2}\,[(k+1)!]^2} \le \frac{1}{(3k+4)1/2}$$

Hence the above inequation is true for $n = k+1$ and by the principle of induction it is true for all $n \in N$.

9. We have $5\sqrt{5}-11 = \dfrac{4}{5\sqrt{5}+11} < 1$

Therefore $0 < 5\sqrt{5}-11 < 1$

This gives us $0 < (5\sqrt{5}-11)^{2n+1} < 1$ for every positive integer n.

Also $(5\sqrt{5}+11)^{2n+1} - (5\sqrt{5}-11)^{2n+1}$

$= 2[{}^{2n+1}C_1(5\sqrt{5})^{2n}.11 + {}^{2n+1}C_3(5\sqrt{5})^{2n-2}.11^3 + \;....+\;{}^{2n+1}C_{2n+1}11^{2n+1}]$

$= 2[{}^{2n+1}C_1(125)^{n}.11 + {}^{2n+1}C_3(125)^{n-1}.11^3 + \;....+\;{}^{2n+1}C_{2n+1}11^{2n+1}]$

$= 2k$ ....(1)

where k is some positive integer.

Let $F = (5\sqrt{5}-11)^{2n+1}$

Then equation (1) becomes

$R - F = 2k$

$\Rightarrow\ [R] + R - [R] - F = 2k \Rightarrow [R] + f - F = 2k$

$\Rightarrow\ f - F = 2k - [R] \Rightarrow f - F$ is an integer.

But $0 \le f < 1$ and $0 < F < 1$ Therefore $-1 < f - F < 1$

Since $f - F$ is an integer, we must have $f - F = 0$

$\Rightarrow\ f = F.$

Now, $Rf = RF = (5\sqrt{5}+11)^{2n+1}(5\sqrt{5}-11)^{2n+1}$

$= [(5\sqrt{5})^2 - 12]^{2n+1} = 4^{2n+1}$

10. Let the given statement be

$P(m,n): {}^mC_0{}^nC_k + {}^mC_1{}^nC_{k-1} + ... + {}^mC_k{}^nC_0 = {}^{m+n}C_k$

where $m,n,k \in N$ and ${}^pC_q = 0$ for $p < q$.

As k is a positive integer and ${}^pC_q = 0$ for $p < q$.

$\therefore$ k must be a positive integer less than or equal to the smaller of m and n,

We have $k = 1$, when $m = n = 1$

$\therefore\ P(1,1)$ is ${}^1C_0{}^1C_1 + {}^1C_1{}^1C_0 = {}^2C_1 \Rightarrow 1+1 = 2.$

Thus $P(1, 1)$ is true.

Now let us assume that $P(m, n)$ holds good for any fixed value of m and n i.e.

$${}^mC_0{}^nC_k + {}^mC_1{}^nC_{k-1} + ... + {}^mC_k{}^nC_0 = {}^{m+n}C_k \quad ....(1)$$

Then $P(m+1,\ n+1)$ will be

$${}^{m+1}C_0{}^{n+1}C_k + {}^{m+1}C_1{}^{n+1}C_{k-1} + ... + {}^{m+1}C_k{}^{n+1}C_0 = {}^{m+n+2}C_k \quad ....(2)$$

Consider LHS

$= {}^{m+1}C_0{}^{n+1}C_k + {}^{m+1}C_1{}^{n+1}C_{k-1} + ... + {}^{m+1}C_k{}^{n+1}C_0$

$= 1.({}^nC_{k-1} + {}^nC_k) + ({}^mC_0 + {}^mC_1)({}^nC_{k-2} + {}^nC_{k-1})$

$+\ ({}^mC_1 + {}^mC_2)({}^nC_{k-3} + {}^nC_{k-2}) + ... + ({}^mC_{k-1} + {}^mC_k).1$

$= ({}^nC_{k-1} + {}^mC_1{}^nC_{k-2} + {}^mC_2\,{}^nC_{k-3} + ... + {}^mC_{k-1}\,{}^nC_0)$

$+ ({}^nC_k + {}^mC_1{}^nC_{k-1} + {}^mC_2\,{}^nC_{k-2} + ... + {}^mC_{k-1}\,{}^nC_1 + {}^mC_k)$

$+ ({}^mC_0\,{}^nC_{k-2} + {}^mC_1\,{}^nC_{k-3} + ... + {}^mC_{k-2}{}^nC_0)$

$+ ({}^mC_0\,{}^nC_{k-1} + {}^mC_1\,{}^nC_{k-2} + {}^mC_2{}^nC_{k-3} + ... + {}^mC_{k-2}{}^nC_1 + {}^mC_{k-1})$

$= {}^{m+n}C_{k-1} + {}^{m+n}C_k + {}^{m+n}C_{k-2} + {}^{m+n}C_{k-1}$ [Using (1)]

$= {}^{m+n+1}C_k + {}^{m+n+1}C_{k-1} = {}^{m+n+2}C_k$

Hence the theorem holds for the next integers $m+1$ and $n+1$. Then by mathematical induction the statement $P(m, n)$ holds for all positive integral values of m and n.

11. We know that

$(1-x)^n = C_0 - C_1x + C_2x^2 - C_3x^3 + ... + (-1)^n\,C_nx^n$

Multiplying both sides by x, we get

$x\,(1-x)^n = C_0x - C_1x^2 + C_2x^3 - C_3x^4 + ... + (-1)^n\,C_nx^{n+1}$

Differentiating both sides w.r. to x, we get

$(1-x)^n - nx\,(1-x)^{n-1}$

$= C_0 - 2C_1x + 3C_2x^2 - 4C_3x^3 + ... + (-1)^n\,(n+1)\,C_nx^n$

Again multiplying both sides by x, we get

$x\,(1-x)^n - nx^2\,(1-x)^{n-1}$

$= C_0x - 2C_1x^2 + 3C_2x^3 - 4C_3x^4 + ... + (-1)^n\,(n+1)\,C_nx^{n+1}$

Differentiating above with respect to x, we get

$(1-x)^n - nx\,(1-x)^{n-1} - 2nx\,(1-x)^{n-1} + nx^2\,(n-1)\,(1-x)^{n-2}$

$= C_0 - 2^2\,C_1x + 3^2\,C_2x^2 - 4^2\,C_3x^3 + ... + (-1)^n\,(n+1)^2\,C_nx^n$

Substituting $x = 1$, in above, we get

$0 = C_0 - 2^2C_1 + 3^2C_2 - 4^2C_3 + ... + (-1)^n\,(n+1)^2C_n$

Hence Proved.

12. We have

$P(n): \dfrac{n^7}{7} + \dfrac{n^5}{5} + \dfrac{2n^3}{3} - \dfrac{n}{105}$ is an integer, $\forall\, n \in N$

$P(1): \dfrac{1}{7} + \dfrac{1}{5} + \dfrac{2}{2} - \dfrac{1}{105}$

$= \dfrac{15+21+70-1}{105} = \dfrac{105}{105} = 1$ an integer

$\therefore$ $P(1)$ is true

Let $P(k)$ be true i.e.

$$\frac{k^7}{7}+\frac{k^5}{5}+\frac{2k^3}{3}-\frac{k}{105} \text{ is an integer}$$

$$\Rightarrow \frac{k^7}{7}+\frac{k^5}{5}+\frac{2k^3}{3}-\frac{k}{105}=m, \text{(say)}$$

$m \in N$ ....(1)

Consider $P(k+1)$:

$$=\frac{(k+1)^7}{7}+\frac{(k+1)^5}{5}+\frac{2(k+1)^3}{3}-\frac{(k+1)}{105}$$

$$=\left(\frac{k^7+7k^6+21k^5+35k^4+35k^3+21k^2+7k+1}{7}\right)$$

$$+\left(\frac{k^5+5k^4+10k^3+10k^2+5k+1}{5}\right)$$

$$+2\left(\frac{k^3+3k^2+3k+1}{3}\right)-\left(\frac{k+1}{105}\right)$$

$$=\left(\frac{k^7}{7}+\frac{k^5}{5}+\frac{2k^3}{3}-\frac{k}{105}\right)$$

$$+[k^6+3k^5+5k^4+5k^3+3k^2+k+k^4$$

$$+2k^3+2k^2+k+2k^2+2k]+\left(\frac{1}{7}+\frac{1}{5}+\frac{2}{3}-\frac{1}{105}\right)$$

$= m +$ some integral value $+ 1$

$=$ some integral value

$\therefore$ $P(k+1)$ is also true.

Hence $P(n)$ is true $\forall\ n \in N$, (by the Principle of Mathematical Induction.)

13. Let $P(k)=\sum_{m=0}^{k}\frac{(n-m)(r+m)!}{m!}=\frac{(r+k+1)!}{k!}\left[\frac{n}{r+1}-\frac{k}{r+2}\right]$

For $k=1$, we will have two terms, on LHS, in sigma for $m=0$ and $m=1$, so that

$$LHS=(n-0)\frac{r!}{0!}+(n-1)\frac{(r+1)!}{1!}$$

and $$RHS=\frac{(r+2)!}{1!}\left[\frac{n}{r+1}-\frac{1}{r+2}\right]$$

Hence LHS = RHS for $k=1$.

Now let the formula holds for $k=s$, that is let

$$\sum_{m=0}^{s}\frac{(n-m)\,(r+m)!}{m!}=\frac{(r+s+1)!}{s!}\left(\frac{n}{r+1}-\frac{s}{r+2}\right) \quad ...(1)$$

Let us add next term corresponding to $m=s+1$ i.e.

adding $\frac{(n-s-1)(r+s+1)!}{(s+1)!}$ to both sides, we get

$$\sum_{m=0}^{s+1}\frac{(n-m)\,(r+m)!}{m!}=\frac{(r+s+1)!}{s!}\left[\frac{n}{r+1}-\frac{s}{r+2}\right]$$

$$+\frac{(n-s-1)(r+s+1)!}{(s+1)!}$$

$$=\frac{(r+s+1)!}{(s+1)!}\left[\frac{(s+1)n}{r+1}-\frac{s(s+1)}{r+2}+n-s-1\right]$$

$$=\frac{(r+s+1)!}{(s+1)!}\left[n\left\{\frac{s+1}{r+1}+1\right\}-(s+1)\left\{\frac{s}{r+2}+1\right\}\right]$$

$$=\frac{(r+s+2)\,(r+s+1)!}{(s+1)!}\left[\frac{n}{r+1}-\frac{s+1}{r+2}\right]$$

Hence the formula holds for $k=s+1$ and so by the induction principle, the formula holds for all natural numbers k.

14. Given that

$$\sum_{r=0}^{2n}a_r(x-2)^r=\sum_{r=0}^{2n}b_r(x-3)^r \quad ....(1)$$

and $a_k=1, \forall\, k \geq n$

To prove $b_n={}^{2n+1}C_{n+1}$

In the given equation (1) let us put $x-3=y$ so that $x-2=y+1$ and we get

$$\sum_{r=0}^{2n}a_r(1+y)^r=\sum_{r=0}^{2n}b_r(y)^r$$

$\Rightarrow$ $a_0+a_1(1+y)+....a_{n-1}+(1+y)^{n-1}(1+y)^n$ $+(1+y)^{n+1}+....(1+y)^{2n}$

$$=\sum_{r=0}^{2n}b_ry^r \quad [\text{Using } a_k=1, \forall\, k \geq n]$$

Equating the coefficients of y^n on both sides we get

NOTE THIS STEP:

$\Rightarrow$ ${}^nC_n+{}^{n+1}C_n+{}^{n+2}C_n+....+{}^{2n}C_n=b_n$

$\Rightarrow$ $({}^{n+1}C_{n+1}+{}^{n+1}C_n)+{}^{n+2}C_n+....+{}^{2n}C_n=b_n$

[Using ${}^nC_n={}^{n+1}C_{n+1}=1$]

$\Rightarrow$ $b_n={}^{n+2}C_{n+1}+{}^{n+2}C_n+....+{}^{2n}C_n$

[Using ${}^mC_r+{}^mC_{r-1}={}^{m+1}C_r$]

Combining the terms in similar way, we get

$\Rightarrow$ $b_n={}^{2n}C_{n+1}+{}^{2n}C_n \Rightarrow b_n={}^{2n+1}C_{n+1}$

Hence Proved

15. Since α, β are the roots of $x^2-(p+1)x+1=0$

$\therefore$ $\alpha+\beta=p+1;\ \alpha\beta=1$

Here $p \geq 3$ and $p \in Z$

(i) To prove that $\alpha^n+\beta^n$ is an integer.

Let us consider the statement, "$\alpha^n+\beta^n$ is an integer."

Then for $n=1$, $\alpha+\beta=p+1$ which is an integer, p being an integer.

$\therefore$ Statement is true for $n=1$

Let the statement be true for $n \leq k$, i.e., $\alpha^k+\beta^k$ is an integer

Then ,

$$\alpha^{k+1}+\beta^{k+1}=\alpha^k.\alpha+\beta^k.\beta$$

$$=\alpha(\alpha^k+\beta^k)+\beta(\alpha^k+\beta^k)-\alpha\beta^k-\alpha^k\beta$$

$$=(\alpha+\beta)(\alpha^k+\beta^k)-\alpha\beta(\alpha^{k-1}+\beta^{k-1})$$

$$=(\alpha+\beta)(\alpha^k+\beta^k)-(\alpha^{k-1}+\beta^{k-1}) \quad .....(1)$$

[as $\alpha\beta=1$]

= difference of two integers = some integral value

$\Rightarrow$ Statement is true for $n=k+1$.

$\therefore$ By the principle of mathematical induction the given statement is true for $\forall\, n \in N$.

(ii) Let R_n be the remainder of $\alpha^n + \beta^n$ when divided by p where $0 \le R_n \le p-1$

Since $\alpha + \beta = p+1 \quad \therefore \quad R_1 = 1$

Also $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = (p+1)^2 - 2$

$= p^2 + 2p - 1 = p\,(p+1) + p - 1$

$\therefore \qquad R_2 = p-1$

Also from equation (1) of previous part (i), we have

$$\alpha^{n+1} + \beta^{n+1} = (p+1)(\alpha^n + \beta^n) - (\alpha^{n-1} + \beta^{n-1})$$

$$= p(\alpha^n + \beta^n) + (\alpha^n + \beta^n) - (\alpha^{n-1} + \beta^{n-1})$$

$\Rightarrow R_{n+1}$ is the remainder of $R_n - R_{n-1}$ when divided by p

$\therefore$ We observe that $R_2 - R_1 = p - 1 - 1$

$\therefore \quad R_3 = p - 2$

Similarly, R_4 is the remainder when $R_3 - R_2$ is divided by p where

$R_3 - R_2 = p - 2 - p + 1 = -1 = -p + (p-1) \quad \therefore \quad R_4 = p-1$

$R_4 - R_3 = p - 1 - p + 1 = 1 \quad \therefore \quad R_5 = 1$

$R_5 - R_4 = 1 - p + 1 = -p + 2 \quad \therefore \quad R_6 = p - 2$

It is evident for above that the remainder is either 1 or $p-1$ or $p-2$.

Since $p \ge 3$, so none is divisible by p.

16. To prove

$$P(n): \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \ldots \tan^{-1}\left(\frac{1}{n^2+n+1}\right)$$

$$= \tan^{-1}\left(\frac{n}{n+2}\right)$$

For $n = 1$, LHS $= \tan^{-1}\frac{1}{3}$;

RHS $= \tan^{-1}\frac{1}{3} \Rightarrow$ LHS = RHS.

$\therefore \quad P(1)$ is true.

Let $P(k)$ be true, i.e.

$$\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \ldots \tan^{-1}\left(\frac{1}{k^2+k+1}\right) = \tan^{-1}\left(\frac{k}{k+2}\right)$$

Consider $P(k+1)$

$$\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} + \ldots \tan^{-1}\left(\frac{1}{k^2+k+1}\right)$$

$$+ \tan^{-1}\left(\frac{1}{(k+1)^2 + (k+1) + 1}\right)$$

$$= \tan^{-1}\left[\frac{k+1}{(k+1)+2}\right]$$

$$\text{LHS} = \tan^{-1}\left[\frac{k}{(k+2}\right] + \tan^{-1}\left(\frac{1}{(k^2+3k+3)}\right)$$

[Using equation (1)]

$$= \tan^{-1}\left[\frac{\frac{k}{(k+2} + \frac{1}{k^2+3k+3}}{1 - \left(\frac{k}{k+2}\right)\left(\frac{1}{k^2+3k+3}\right)}\right]$$

$$= \tan^{-1}\left[\frac{(k+1)(k^2+2k+2)}{(k+3)(k^2+2k+2)}\right] = \tan^{-1}\left(\frac{k+1}{k+3}\right) = RHS$$

$\therefore \quad P(k+1)$ is also true.

Hence by the principle of mathematical induction $P(n)$ is true for every natural number.

17. To evaluate $\sum_{r=1}^{k} (-3)^{r-1}\,{}^{3n}C_{2r-1}$ where $k = \frac{3n}{2}$ and n is +ve even interger.

Let $n = 2m$, where $m \in z^+ \quad \therefore \; k = \frac{3(2m)}{2} = 3m$

$$\therefore \quad \sum_{r=1}^{k} (-3)^{r-1}\,{}^{3n}C_{2r-1} = \sum_{r=1}^{3m} (-3)^{r-1}\,{}^{6m}C_{2r-1}$$

$$= {}^{6m}C_1 - 3.{}^{6m}C_3 + 3^2\,{}^{6m}C_5 - \ldots\ldots \qquad \ldots(1)$$

Now we know that

$(1+a)^{6m} - (1-a)^{6m}$

$= 2[{}^{6m}C_1 a + {}^{6m}C_3 a^3 + {}^{6m}C_5 a^5 + \ldots..] \qquad \ldots(2)$

Keeping in mind the form of RHS in equation (1) and in equation (2)

We put $a = i\sqrt{3}$ in equation (2) to get

$(1+i\sqrt{3})^{6m} - (1+i\sqrt{3})^{6m}$

$= 2[{}^{6m}C_1 i\sqrt{3} - {}^{6m}C_3 i3\sqrt{3} + {}^{6m}C_5 i3^2\sqrt{3}\ldots.]$

$\Rightarrow \quad (1+i\sqrt{3})^{6m} - (1-i\sqrt{3})^{6m}$

$= 2\sqrt{3}\,i[{}^{6m}C_1 - 3.{}^{6m}C_3 + 3^2\,{}^{6m}C_5\ldots.]\ldots.(3)$

But $1 + i\sqrt{3} = 2(\cos\pi/3 + i\sin\pi/3)$

$\therefore \quad (1+i\sqrt{3})^{6m} = 2^{6m}(\cos\pi/3 + i\sin\pi/3)^{6m}$

NOTE THIS STEP

$= 2^{6m}\left(\cos\frac{6m\pi}{3} + i\sin\frac{6m\pi}{3}\right)$ [Using D' Moivre's thm.]

Similarly,

$(1-i\sqrt{3})^{6m} = 2^{6m}\left(\cos\frac{6m\pi}{3} - i\sin\frac{6m\pi}{3}\right)$

$\therefore \quad (1+i\sqrt{3})^{6m} - (1-i\sqrt{3})^{6m} = 2^{6m}.2\sin 2m\pi = 0$

Substituting the above in equation (3) we get

${}^{6m}C_1 - 3.{}^{6m}C_3 + 3^2\,{}^{6m}C_5 - \ldots.. = 0$

$\Rightarrow \quad \sum_{r=1}^{k} (-3)^{r-1}\,{}^{3n}C_{2r-1} = 0.$

Hence Proved

18. Let $P(n): \cos x + \cos 2x + \ldots + \cos nx$

$= \cos\frac{n+1}{2}x \sin\frac{nx}{2} \operatorname{cosec}\frac{x}{2} \qquad \ldots.(1)$

where x is not an integral multiple of 2π.

For $n = 1$ $P(1)$: L.H.S. $= \cos x$

$R.H.S. = \cos\frac{1+1}{2}x \sin\frac{x}{2} \operatorname{cosec}\frac{x}{2} = \cos x$

L.H.S. = R.H.S.

$\Rightarrow \quad P(1)$ is true.

Let $P(k)$ be true i.e.
$\cos x + \cos 2x + ... + \cos kx$

$= \cos\frac{k+1}{2}x \sin\frac{kx}{2} \text{ cosec } \frac{x}{2}$ ....(2)

Consider $P(k+1)$:
$\cos x + \cos 2x + ... + \cos kx + \cos(k+1)x$

$= \cos\left(\frac{k+2}{2}\right)x \sin\frac{(k+1)x}{2} \text{ cosec } \frac{x}{2}$

L.H.S. $[\cos x + \cos 2x + ... + \cos kx + \cos(k+1)x$

$= \cos\left(\frac{k+1}{2}\right)x \sin \text{cosec}\frac{kx}{2}\ \frac{x}{2} + \cos(k+1)x$

[Using (2)]

$= \left[\cos\left(\frac{k+1}{2}\right)x \sin\frac{kx}{2} + \cos(k+1)x \sin\frac{x}{2}\right]\text{cosec } \frac{x}{2}$

$= \frac{1}{2}\left[2\cos\frac{(k+1)x}{2}\sin\frac{kx}{2} + 2\cos(k+1)x\sin\frac{x}{2}\right]\text{cosec } \frac{x}{2}$

$= \frac{1}{2}[\sin\left(\frac{2k+1}{2}\right)x - \sin\frac{x}{2}$

$+ \sin\left(xk + \frac{3x}{2}\right) - \sin\left(xk + \frac{x}{2}\right)]\text{cosec } \frac{x}{2}$

$= \frac{1}{2}\left[\sin\left(xk + \frac{3x}{2}\right) - \sin\frac{x}{2}\right]\text{cosec } \frac{x}{2}$

$= \frac{1}{2}\left[2\cos\frac{(k+2)x}{2}\sin\frac{(k+1)}{2}\right]\text{cosec } \frac{x}{2}$

$= \cos\frac{(k+2)x}{2}\sin\frac{(k+1)x}{2}\text{cosec } \frac{x}{2} = R.H.S.$

$\therefore$ $P(k+1)$ is also true.
Hence by the principle of mathematical induction
$P(n)$ is true $\forall n \in N$.

19. Given that,
$(1+x+x^2)^n = a_0 + a_1x + ... + a_{2n}x^{2n}$ ....(1)
where n is a +ve integer.

Replacing x by $-\frac{1}{x}$ in eq^n (1), we get

$\left(1 - \frac{1}{x} + \frac{1}{x^2}\right)^n = a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} - \frac{a_3}{x^3} + ... + \frac{a_{2n}}{x^{2n}}$ ....(2)

Multiplying eq.'s (1) and (2):

$\frac{(1+x+x^2)^n (x^2 - x + 1)^n}{x^{2n}}$

$= (a_0 + a_1x + ... + a_{2n}x^{2n})\ (a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} + ... + \frac{a_n}{x^{2n}}$

Equating the constant terms on both sides we get

$a_0^2 - a_1^2 + a_2^2 - a_3^2 + ... + a_{2n}^2 =$ constant term in the expansion of $\frac{[(1+x+x^2)(1-x+x^2)]^n}{x^{2n}}$

= Coeff. of x^{2n} in the expansion of $(1+x^2+x^4)^n$
But replacing x by x^2 in eq's (1), we have

$(1+x^2+x^4)^n = a_0 + a_1x^2 + ... + a_{2n}(x^2)^{2n}$
$\therefore$ Coeff of $x^{2n} = a_n$

Hence we obtain, $a_0^2 - a_1^2 + a_2^2 - a_3^2 + ... + a_{2n}^2 = a_n$

20. For $n = 1$, $3^{2^n} - 1 = 3^{2^1} - 1 = 9 - 1 = 8$ which is divisible by $2^{n+2} = 2^3 = 8$ but is not divisible by $2^{n+3} = 2^4 = 16$
Therefore, the result is true for $n = 1$.
Assume that the result is true for $n = k$. That is, assume that

$3^{2^k} - 1$ is divisible by 2^{k+2} but is not divisible by 2^{k+3},

Since $3^{2^k} - 1$ is divisble by 2^{k+2} but not by 2^{k+3}, we can write $3^{2^k} - 1 = (m)2^{k+2}$
where m must be an odd positive integer, for otherwise $3^{2k} - 1$ will become divisible by 2^{k+3}.

For $n = k+1$, we have $3^{2^{k+1}} - 1 = 3^{2^k.2} - 1 = \left(3^{2^k}\right)^2 - 1$
$= (m.2^{k+2} + 1)^2 - 1$ [Using (1)]
$= m^2.(2^{k+2})^2 + 2m.2^{k+2} + 1 - 1$
$= m^2.2^{2k+4} + m.2^{k+3} = 2^{k+3}(m^2.2^{k+1} + m.)$

$\Rightarrow$ $3^{2^{k+1}} - 1$ is divisible by 2^{k+3}.

But $3^{2^{k+1}} - 1$ is not divisible by 2^{k+4} for otherwise we must have 2 divides $m^2.\ 2^{k+1} + m$. But this is not possible as m is odd. Thus, the result is true for $n = k+1$.

21. For $n = 1$, the inequality becomes
$\sin A_1 \le \sin A_1$, which is clearly true.
Assume that the inequality holds for $n = k$ where k is some positive integer. That is, assume that

$\sin A_1 + \sin A_2 + ... + \sin A_k \le k \sin\left(\frac{A_1 + A_2 + ... + A_k}{k}\right)$ ....(1)

for same positive integer k.
We shall now show that the result holds for $n = k+1$ that is, we show that

$\sin A_1 + \sin A_2 + ... + \sin A_k + \sin A_{k+1}$

$\le (k+1)\sin\left(\frac{A_1 + A_2 + ... + A_{k+1}}{k+1}\right)$ ....(2)

L.H.S. of (2)
$= \sin A_1 + \sin A_2 + ... + \sin A_k + \sin A_{k+1}$

$\le k\sin\left(\frac{A_1 + A_2 + ... + A_k}{k}\right) + \sin A_{k+1}$

[Induction assumption]

$= (k+1)\left[\frac{k}{k+1}\sin\alpha + \frac{1}{k+1}\sin A_{k+1}\right];$

where $\alpha + \frac{A_1 + A_2 + ...A_k}{k}$

$\therefore$ L.H.S. of (2) $\le (k+1)\left[\left(1 - \frac{k}{k+1}\right)\sin\alpha + \frac{1}{k+1}\sin A_{k+1}\right]$

$\le (k+1)\sin\left\{\left(1 - \frac{k}{k+1}\right)\alpha + \frac{1}{k+1}A_{k+1}\right\}$

[Using the fact $p\sin x + (1-p)\sin y \le \sin[px + (1-p)y]$ for $0 \le p \le 1, 0 \le x, y \le \pi$]

$= (k+1)\sin\left\{\frac{k}{k+1}\left(\frac{A_1+A_2+...A_k}{k}\right)+\frac{1}{k+1}A_{k+1}\right\}$

$= (k+1)\sin\left(\frac{A_1+A_2+....A_{k+1}}{k+1}\right)$

Thus, the inequality holds for $n = k + 1$. Hence, by the principle of mathematical induction the inequality holds for all $n \in N$.

22. We know that ${}^nC_r = \frac{n}{r}\,{}^{n-1}C_{r-1}$

$\therefore \quad {}^{mp}C_r = \frac{mp}{r}\,{}^{mp-1}C_{r-1} = \left[\frac{m.{}^{mp-1}C_{r-1}}{r}\right]p$

Now, L.H.S is an integer

$\Rightarrow$ RHS must be an integer

But p and r are coprime (given)

$\therefore$ r must divide $m.\,{}^{mp-1}C_{r-1}$

or $\frac{m.\,{}^{mp-1}C_{r-1}}{r}$ is an integer.

$\Rightarrow \frac{{}^{mp}C_r}{p}$ is an integer or ${}^{mp}C_r$ is divisible by p.

23. Let $P(m) = \sum_{k=0}^{m}\frac{\binom{2n-k}{k}^{(2n-4k+1)}}{\binom{2n-k}{n}^{(2n-2k+1)}}2^{n-2k}$

$= \frac{\binom{n}{m}}{\binom{2n-2m}{n-m}}.2^{n-2m}$ ....(1)

For $m=0$, LHS $= \frac{\binom{2n}{0}}{\binom{2n}{n}}\cdot\frac{2n+1}{2n+1}\,.\,2^n = \frac{1}{\binom{2n}{n}}2^n$,

R.H.S. $= \frac{\binom{n}{0}}{\binom{2n}{n}}\,.\,2^n = \frac{1}{\binom{2n}{n}}2^n = L.H.S$

$[\because m=0 \Rightarrow k=0]$

$\therefore$ $P(0)$ holds true. Now assuming $P(m)$

L.H.S. of $P(m+1)=$ L.H.S. of

$P(m)+\frac{\binom{2n-m-1}{m+1}}{\binom{2n-m-1}{n}}\cdot\frac{(2n-4m-3)}{(2n-2m-1)}.2^{n-2m-2}$

$= \frac{n!(n-m)!}{m!(2n-2m)!}.2^{n-2m}$

$+ \frac{n!(n-m-1)!(2n-4m-3)}{(m+1)!(2n-2m-2)!(2n-2m-1)}.2^{n-2m-2}$

$= \frac{n!(n-m-1)!2^{n-2m-2}}{(m+1)!(2n-2m-1)!}$

$\times\left\{\frac{(n-m).4\,(m+1)}{(2n-2m)}+(2n-4m-3)\right\}$

$= \frac{n!(n-m-1)!2^{n-2m-2}\,(2n-2m-1)}{(m+1)!(2n-2m-1)!}$

$= \frac{n!(n-m-1)!2^{n-2m-2}}{(m+1)!(2n-2m-2)!} = \frac{\binom{n}{m+1}}{\binom{2n-2m-2}{n-m-1}}.2^{n-2m-2}$

$=$ R.H.S. of $P(m+1)$.

Hence by mathematical induction, result follows for all $0 \le m \le n$.

24. Given that for positive integers m and n such that $n \ge m$, then to prove that

${}^nC_m + {}^{n-1}C_m + {}^{n-2}C_m + ... + {}^mC_m = {}^{n+1}C_{m+1}$

L.H.S. ${}^mC_m + {}^{m+1}C_m + {}^{m+2}C_m + ... + {}^{n-1}C_m + {}^nC_m$

[writing L.H.S. in reverse order]

$= ({}^{m+1}C_{m+1} + {}^{m+1}C_m) + {}^{m+2}C_m + ... + {}^{n-1}C_m + {}^nC_m$

$[\because {}^mC_m = {}^{m+1}C_{m+1}]$

$= ({}^{m+2}C_{m+1} + {}^{m+2}C_m) + {}^{m+3}C_m + ... + {}^nC_m$

$[\because {}^nC_{r+1} + {}^nC_r = {}^{n+1}C_{r+1}]$

$= {}^{m+3}C_{m+1} + {}^{m+3}C_m + ... + {}^nC_m$

Combining in the same way we get

$= {}^nC_{m+1} + {}^nC_m = {}^{n+1}C_{m+1} =$ R.H.S.

Again we have to prove

${}^nC_m + 2\,{}^{n-1}C_m + 3\,{}^{n-2}C_m + ... + (n-m+1)\,{}^mC_m$

$= {}^{n+2}C_{m+2}$

$= [{}^nC_m + {}^{n-1}C_m + {}^{n-2}C_m + ... + {}^mC_m] + [{}^{n-1}C_m + {}^{n-2}C_m + ... + {}^mC_m] + [{}^{n-2}C_m + ... + {}^mC_m] + ... + [{}^mC_m]$

$[n-m+1$ bracketed terms]

$= {}^{n+1}C_{m+1} + {}^nC_{m+1}\,{}^{n-1}C_{m+1} .... + {}^{m+1}C_{m+1}$

[using previous result.]

$= {}^{n+2}C_{m+2}$

[Replacing n by $n+1$ and m by $m+1$ in the previous result.]

$=$ R.H.S.

25. For $n > 0, \sqrt{4n+1} > 0, \sqrt{n}+\sqrt{n+1} > 0$ and $\sqrt{4n+2} > 0$

Now, $\sqrt{4n+1} < \sqrt{n}+\sqrt{n+1} < \sqrt{4n+2}$ to be proved.

I. To prove $\sqrt{4n+1} < \sqrt{n}+\sqrt{n+1}$

Squaring both sides in $\sqrt{4n+1} < \sqrt{n}+\sqrt{n+1}$

$\Rightarrow \quad 4n+1 < n+n+1+2\sqrt{n(n+1)}$

$\Rightarrow \quad 2n < 2\sqrt{n(n+1)} \Rightarrow n < \sqrt{n(n+1)}$ which is true.

II. To prove $\sqrt{n}+\sqrt{n+1} < \sqrt{4n+2}$

Squaring both sides,

$n+n+1+2\sqrt{n(n+1)} < 4n+2$

$\Rightarrow \quad 2\sqrt{n(n+1)} < 2n+1$ Squaring again

$4\,[n\,(n+1)] < 4n^2+1+4n$ or $0 < 1$ which is true

Hence $\sqrt{4n+1} < \sqrt{n}+\sqrt{n+1} < \sqrt{4n+2}$

Further to prove $[\sqrt{n}+\sqrt{n+1}]=[\sqrt{4n+1}]$, we have to prove that there is no positive integer which lies between $\sqrt{4n+1}$ and $\sqrt{4n+2}$ or $[\sqrt{4n+1}]=[\sqrt{4n+2}]$. Using Mathematical induction.

We have to check $[\sqrt{4n+1}]=[\sqrt{4n+2}]$ for $n=1$

$[\sqrt{5}]=[\sqrt{6}] \Rightarrow 2=2$, which is true

Assume for $n=$ k (arbitrary)

i.e., $[\sqrt{4k+1}]=[\sqrt{4k+2}]$ To prove for $n=k+1$

To check $[\sqrt{4k+5}]=[\sqrt{4k+6}]$ since $k \geq 0$

Here $4k+5$ is an odd number and $4k+6$ is even number. Their greatest integer will be different iff $4k+6$ is a perfect square that is $4k+6=r^2$

$\Rightarrow \quad k=\frac{r^2}{4}-\frac{6}{4}, \frac{6}{4}$ is not integer. But k has to be integer.

So 4k + 6 cannot be perfect square.

$\Rightarrow \quad [\sqrt{4k+5}]=[\sqrt{4k+6}]$

By Sandwich theorem

$\Rightarrow \quad [\sqrt{n}+\sqrt{n+1}]=[\sqrt{4n+1}]$

26. We have a, b, c the +ve real number s.t. $b^2-4ac>0$; $\alpha_1=$ c.

$$P(n): \alpha_{n+1}=\frac{a\alpha_n^2}{b^2-2a(\alpha_1+\alpha_2+....\alpha_n)}$$

is well defined and $\alpha_{n+1}<\frac{\alpha_n}{2}, \forall\, n=1,2,....$

For $n=1$, $\alpha_2=\frac{a\alpha_1^2}{b^2-2a\alpha_1}=\frac{ac^2}{b^2-2ac}$

Now, $b^2-4ac>0 \Rightarrow b^2-2ac>2ac>0$

$\therefore \quad \alpha_2$ is well defined (as denomination is not zero)

$$\text{Also}\left[\begin{array}{l}\because \quad b^2-2ac>2ac \\ \Rightarrow \quad \frac{1}{b^2-2ac}<\frac{1}{2ac}\end{array}\right] \Rightarrow \frac{\alpha_2}{c}<\frac{1}{2} \Rightarrow \frac{\alpha_2}{\alpha_1}<\frac{1}{2}$$

$\therefore \quad P(n)$ is true for $n=1$.

Let the statement be true for $1 \leq n \leq k$ i.e.,

$$\alpha_{k+1}=\frac{a\alpha_k^2}{b^2-2a(\alpha_1+\alpha_2+....\alpha_k)} \text{ is well defined}$$

and $\alpha_{k+1}<\frac{\alpha_k}{2}$

Now, we will prove that $P(k+1)$ is also true

$$\text{i.e., } \alpha_{k+2}=\frac{a\alpha_{k+1}^2}{b^2-2a(\alpha_1+\alpha_2+....+\alpha_k+\alpha_{k+1})} \text{ is}$$

well defined and $\alpha_{k+2}<\frac{\alpha_{k+1}}{2}$.

We have

$$\alpha_1=c, \alpha_2<\frac{c}{2}, \alpha_3<\frac{\alpha_2}{2}<\frac{c}{2^2}, \alpha_4<\frac{\alpha_3}{2}<\frac{c}{2^3}, ... \text{ (by IH)}$$

Now, $(\alpha_1+\alpha_2+...\alpha_k+\alpha_{k+1}<c+\frac{c}{2}+\frac{c}{2^k}+....+\frac{c}{2^k}$

$$=\frac{c\left(1-\frac{1}{2^{k+1}}\right)}{1-1/2}=2c\left(1-\frac{1}{2^{k+1}}\right)<2c$$

$\therefore \quad \alpha_1+\alpha_2+...+\alpha_{k+1}<2c$

$\Rightarrow \quad -2a(\alpha_1+\alpha_2+......+\alpha_{k+1})>-4ac$

$\Rightarrow \quad b^2-2a(\alpha_1+\alpha_2+......+\alpha_{k+1})>b^2-4ac>0$

$\therefore \quad \alpha_{k+2}$ is well defined. Again by IH we have

$$\alpha_{k+1}<\frac{\alpha_k}{2} \Rightarrow 2\alpha_{k+1}<\alpha_k$$

$\Rightarrow \quad 4\alpha_{k+1}^2<\alpha_k^2$ [As by def. α_{k+1}, α_k are +ve]

$\Rightarrow \quad 4\alpha_{k+1}<\frac{\alpha_k^2}{\alpha_{k+1}}$

$\Rightarrow \quad 4\alpha_{k+1}<\frac{b^2-2a(\alpha_1+\alpha_2+....+\alpha_k)}{a}$

$\Rightarrow \quad 4a\alpha_{k+1}<b^2-2a(\alpha_1+\alpha_2+....+\alpha_k)$

$\Rightarrow \quad 2a\alpha_{k+1}<b^2-2a(\alpha_1+\alpha_2+....+\alpha_k+\alpha_{k+1})$

$\Rightarrow \quad \frac{a\alpha_{k+1}^2}{b^2-2a(\alpha_1+\alpha_2+....+\alpha_{k+1})}<\frac{1}{2}$

$\Rightarrow \quad \frac{a\alpha_{k+1}}{b^2-2a(\alpha_1+\alpha_2+....+\alpha_{k+1})}<\frac{\alpha_{k+1}}{2}$

$\Rightarrow \quad \alpha_{k+2}<\frac{\alpha_{k+1}}{2}$

$\therefore \quad P(k+1)$ is also true.

Thus by the Principle of Mathematical Induction the Statement $P(n)$ is true $\forall\, n \in N$.

27. Let $P(n): (25)^{n+1}-24n+5735$

For n = 1.

P(1) : $625-24+5735=6336=(24)^2 \times (11)$, which is divisible by 24^2. Hence P(1) is true

Let $P(k)$ be true, where $k \geq 1$

$\Rightarrow \quad (25)^{k+1}-24k+5735=(24)^2 \lambda$ where $\lambda \in N$

For $n=k+1, P(k+1): (25)^{k+2}-24(k+1)+5735$

$=25[(25)^{k+1}-24k+5735]$

$+25.24.k-(25)(5735)+5735-24(k+1)$

$=25(24)^2 \lambda+(24)^2 k-5735 \times 24-24$

$=25(24)^2 \lambda+(24)^2 k-(24)(5736)$

$=25(24)^2 \lambda+(24)^2 k-(24)^2(239)$,

$=(24)^2[25\lambda+k-239]$ which is divisible by $(24)^2$.

Hence, by the method of mathematical induction result is true $\forall\, n \in N$.

28. To prove that

$$2^k\, {}^nC_0\, {}^nC_k-2^{k-1}\, {}^nC_1\, {}^{n-1}C_{k-1}+2^{k-2}\, {}^nC_2\, {}^{n-2}C_{k-2}$$
$$-.....+(-1)^k\, {}^nC_k\, {}^{n-k}C_0={}^nC_k$$

LHS of above equation can be written as

$$\sum_{r=0}^{k}(-1)^r 2^{k-r}\, {}^nC_r\, {}^{n-r}C_{k-r}$$

$$=\sum_{r=0}^{k}(-1)^r 2^{k-r}\frac{n!}{r!(n-r)!}\frac{(n-r)!}{(k-r)!(n-k)!}$$

$$=\sum_{r=0}^{k}(-1)^r 2^{k-r}\frac{n!k!}{r!k!(n-k)!(k-r)!}$$

$$=\sum_{r=0}^{k}(-1)^r \frac{2^k}{2^r}\cdot\frac{n!}{k!(n-k)!}\frac{k!}{r!(k-r)!}$$

$$=2^k\, {}^nC_k\sum_{r=0}^{k}(-1/2)^r\frac{k!}{r!(k-r)!}$$

$$=2^k\, {}^nC_k\sum_{r=0}^{k}{}^kC_r(-1/2)^r=2^k\, {}^nC_k(1-1/2)^k$$

$$=2^k\, {}^nC_k\frac{1}{2^k}={}^nC_k=\text{R.H.S.}$$ **Hence Proved**

29. We have $\alpha+\beta=1-p$ and $\alpha\beta=-p(1-p)$

For $n=1, p_n=p_1=1$

Also, $A\alpha^n+B\beta^n=A\alpha+B\beta=\dfrac{(p^2+\beta-1)\alpha}{\alpha\beta-\alpha^2}$

$$+\frac{(p^2+\alpha-1)\beta}{\alpha\beta-\beta^2}=\frac{p^2+\beta-1}{\beta-\alpha}+\frac{p^2+\alpha-1}{\alpha-\beta}$$

$$=\frac{p^2+\beta-1-p^2-\alpha+1}{\beta-\alpha}=\frac{\beta-\alpha}{\beta-\alpha}=1$$

For $n=2, p_2=1-p^2$

Also, $A\alpha^n+B\beta^n=A\alpha^2+B\beta^2$

$$=\frac{(p^2+\beta-1)\alpha^2}{\alpha\beta-\alpha^2}+\frac{(p^2+\alpha-1)\beta^2}{\alpha\beta-\alpha^2}$$

which is true for $n=2$

Now let result is true for $k<n$ where $n\ge 3$.

$P_n=(1-p)P_{n-1}+p(1-p)P_{n-2}$

$=(1-p)(A\alpha^{n-1}+B\beta^{n-1})+p(1-p)(A\alpha^{n-2}+B\beta^{n-2})$

$=A\alpha^{n-2}\{(1-p)\alpha+p(1-p)\}+B\beta^{n-2}\{(1-p)\beta-p(1-p)\}$

$=A\alpha^{n-2}\{(\alpha+\beta)\alpha-\alpha\beta\}$

$+B\beta^{n-2}\{(\alpha+\beta)\beta-\alpha\beta\}$ by (1)

$=A\alpha^{n-2}\{\alpha^2+\beta\alpha-\alpha\beta\}+B\beta^{n-2}\{\alpha\beta+\beta^2-\alpha\beta\}$

$=A\alpha^{n-2}(\alpha^2)+B\beta^{n-2}(\beta^2)=A\alpha^n+B\beta^n$

This is true for n. Hence by principle of mathematical induction, the result holds good for all $n\in N$.

I. Integer Value Correct Type

1. **(6)** Let the coefficients of three consecutive terms of $(1+x)^{n+5}$ be

${}^{n+5}C_{r-1}, {}^{n+5}C_r, {}^{n+5}C_{r+1}$, then we have

${}^{n+5}C_{r-1}:{}^{n+5}C_r:{}^{n+5}C_{r+1}=5:10:14$

$$\frac{{}^{n+5}C_{r-1}}{{}^{n+5}C_r}=\frac{5}{10}\Rightarrow\frac{r}{n+6-r}=\frac{1}{2}$$

or $n-3r+6=0$ ...(1)

Also $\dfrac{{}^{n+1}C_r}{{}^{n+5}C_{r+1}}=\dfrac{10}{14}\Rightarrow\dfrac{r+1}{n-r+5}=\dfrac{5}{7}$

or $5n-12r+18=0$ (2)

Solving (1) and (2) we get $n=6$.

2. **(5)** $(1+x)^2+(1+x)^3+....+(1+x)^{49}+(1+mx)^{50}$

$$=(1+x)^2\left[\frac{(1+x)^{48}-1}{(1+x)-1}\right]+(1+mx)^{50}$$

$$=\frac{1}{x}\left[(1+x)^{50}-(1+x)^2\right]+(1+mx)^{50}$$

Coeff. of x^2 in the above expansion

$=$ Coeff. of x^3 in $(1+x)^{50}$ + Coeff. of x^2 in $(1+mx)^{50}$

$\Rightarrow {}^{50}C_3+{}^{50}C_2m^2$

$\therefore (3n+1)\,{}^{51}C_3={}^{50}C_3+{}^{50}C_2m^2$

$$\Rightarrow(3n+1)=\frac{{}^{50}C_3}{{}^{51}C_3}+\frac{{}^{50}C_2}{{}^{51}C_3}m^2$$

$$\Rightarrow 3n+1=\frac{16}{17}+\frac{1}{17}m^2\Rightarrow n=\frac{m^2-1}{51}$$

Least positive integer m for which n is an integer is m = 16 and then n = 5

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Section-B JEE Main/ AIEEE

1. (a) We have $t_{p+1} = {}^{p+q}C_p x^p$ and $t_{q+1} = {}^{p+q}C_q x^q$
${}^{p+q}C_p = {}^{p+q}C_q$.[Remember ${}^nC_r = {}^nC_{n-r}$]

2. (c) We have $2^n = 4096 = 2^{12} \Rightarrow n = 12$; the greatest coeff = coeff of middle term. So middle term

$= t_7$.; $t_7 = t_{6+1} \Rightarrow$ coeff of $t_7 = {}^{12}C_6 = \frac{12!}{6!6!} = 924.$

3. (d) $(1+0.0001)^{10000} = \left(1+\frac{1}{n}\right)^n, n = 10000$

$$= 1 + n.\frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{1}{n^3} + \ldots\ldots$$

$$= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right) + \left(1-\frac{2}{n}\right) + \ldots\ldots$$

$$< 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots\ldots + \frac{1}{(9999)!}$$

$$= 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots\ldots\infty = e < 3$$

4. (c) $t_{r+2} = {}^{2n}C_{r+1}\, x^{r+1}; t_{3r} = {}^{2n}C_{3r-1}\, x^{3r-1}$
Given ${}^{2n}C_{r+1} = {}^{2n}C_{3r-1}$;
$\Rightarrow {}^{2n}C_{2n-(r+1)} = {}^{2n}C_{3r-1}$
$\Rightarrow 2n - r - 1 = 3r - 1 \Rightarrow 2n = 4r \Rightarrow n = 2r$

5. (b) $a_1 = \sqrt{7} < 7$. Let $a_m < 7$

Then $a_{m+1} = \sqrt{7 + a_m} \Rightarrow a^2_{m+1} = 7 + a_m < 7 + 7 < 14.$

$\Rightarrow a_{m+1} < \sqrt{14} < 7$; So by the principle of mathematical induction $a_n < 7\ \forall\ n$.

6. (d) $T_{r+1} = \frac{n(n-1)(n-2)\ldots\ldots\ldots(n-r+1)}{r!}(x)^r$

For first negative term, $n - r + 1 < 0 \Rightarrow r > n + 1$

$\Rightarrow r > \frac{32}{5}\ \therefore r = 7.\ \left(\because n = \frac{27}{5}\right)$

Therefore, first negative term is T_8.

7. (c) $T_{r+1} = {}^{256}C_r(\sqrt{3})^{256-r}(\sqrt[8]{5})^r = {}^{256}C_r(3)^{\frac{256-r}{2}}(5)^{r/8}$

Terms will be integral if $\frac{256-r}{2}$ & $\frac{r}{8}$ both are +ve integer, which is so if r is an integral multiple of 8. As $0 \le r \le 256$

$\therefore$ r = 0,8,16,24,........256 , total 33 values.

8. (b) $S(k) = 1+3+5+\ldots+(2k-1) = 3 + k^2$
$S(1): 1 = 3 + 1$, which is not true
$\because S(1)$ is not true.
$\therefore$ P.M.I cannot be applied
Let $S(k)$ is true, i.e.

$$1 + 3 + 5\ldots. + (2k-1) = 3 + k^2$$

$$\Rightarrow 1 + 3 + 5\ldots. + (2k-1) + 2k + 1$$

$$= 3 + k^2 + 2k + 1 = 3 + (k+1)^2$$

$\therefore S(k) \Rightarrow S(k+1)$

9. (c) The middle term in the expansion of
$(1+\alpha x)^4 = T_3 = {}^4C_2(\alpha x)^2 = 6\alpha^2x^2$
The middle term in the expansion of
$(1-\alpha x)^6 = T_4 = {}^6C_3(-\alpha x)^3 = -20\alpha^3x^3$
According to the question

$$6\alpha^2 = -20\alpha^3 \Rightarrow \alpha = -\frac{3}{10}$$

10. (b) Coeff of x^n in $(1+x)(1-x)^n$

$=$ Coeff of x^n in $(1-x)^n$ + Coeff of x^{n-1} in $(1-x)^n$

$= (-1)^n\, {}^nC_n + (-1)^{n-1}\, {}^nC_{n-1} = (-1)^n 1 + (-1)^{n-1} n$

$= (-1)^n [1 - n]$

11. (d) ${}^{50}C_4 + \sum_{r=1}^{6} {}^{56-r}C_3$

$$\Rightarrow {}^{50}C_4 + \left[\begin{matrix} {}^{55}C_3 + {}^{54}C_3 + {}^{53}C_3 + {}^{52}C_3 \\ + {}^{51}C_3 + {}^{50}C_3 \end{matrix}\right]$$

We know $\left[{}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r\right]$

$$\Rightarrow ({}^{50}C_4 + {}^{50}C_3)$$
$$+ {}^{51}C_3 + {}^{52}C_3 + {}^{53}C_3 + {}^{54}C_3 + {}^{55}C_3$$

$$\Rightarrow ({}^{51}C_4 + {}^{51}C_3) + {}^{52}C_3 + {}^{53}C_3 + {}^{54}C_3 + {}^{55}C_3$$

Proceeding in the same way, we get

$$\Rightarrow {}^{55}C_4 + {}^{55}C_3 = {}^{56}C_4.$$

12. (a) We observe that

$A^2 = \begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}, A^3 = \begin{bmatrix}1 & 0\\ 3 & 1\end{bmatrix}$ and we can prove by induction that $A^n = \begin{bmatrix}1 & 0\\ n & 1\end{bmatrix}$

Now $nA-(n-1)I=\begin{bmatrix} n & 0 \\ n & n \end{bmatrix}-\begin{bmatrix} n-1 & 0 \\ 0 & n-1 \end{bmatrix}$

$=\begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}=A^n$

$\therefore nA-(n-1)I=A^n$

13. **(d)** T_{r+1} in the expansion

$$\left[ax^2+\frac{1}{bx}\right]^{11}={}^{11}C_r(ax^2)^{11-r}\left(\frac{1}{bx}\right)^r$$

$={}^{11}C_r\,(a)^{11-r}(b)^{-r}(x)^{22-2r-r}$

For the Coefficient of x^7, we have

$\Rightarrow 22-3r=7 \Rightarrow r=5$

$\therefore$ Coefficient of $x^7={}^{11}C_5(a)^6\,(b)^{-5}$ ...(1)

Again T_{r+1} in the expansion

$$\left[ax-\frac{1}{bx^2}\right]^{11}={}^{11}C_r(ax^2)^{11-r}\left(-\frac{1}{bx^2}\right)^r$$

$={}^{11}C_r(a)^{11-r}(-1)^r\times(b)^{-r}(x)^{-2r}(x)^{11-r}$

For the Coefficient of x^{-7}, we have

Now $11-3r=-7 \Rightarrow 3r=18 \Rightarrow r=6$

$\therefore$ Coefficient of $x^{-7}={}^{11}C_6\,a^5\times 1\times(b)^{-6}$

$\therefore$ Coefficient of x^7 = Coefficient of x^{-7}

$\Rightarrow {}^{11}C_5\,(a)^6\,(b)^{-5}={}^{11}C_6\,a^5\times(b)^{-6} \Rightarrow ab=1.$

14. **(c)** $\because x^3$ and higher powers of x may be neglected

$$\therefore \frac{(1+x)^{\frac{3}{2}}-\left(1+\frac{x}{2}\right)^3}{\left(1-x\right)^{\frac{1}{2}}}$$

$$=(1-x)^{\frac{-1}{2}}\left[\left(1+\frac{3}{2}x+\frac{\frac{3}{2}\cdot\frac{1}{2}}{2!}x^2\right)-\left(1+\frac{3x}{2}+\frac{3.2}{2!}\frac{x^2}{4}\right)\right]$$

$$=\left[1+\frac{x}{2}+\frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}x^2\right]\left[\frac{-3}{8}x^2\right]=\frac{-3}{8}x^2$$

(as x^3 and higher powers of x can be neglected)

15. **(d)** $(1-ax)^{-1}(1-bx)^{-1}$

$=(1+ax+a^2x^2+...)(1+bx+b^2x^2+...)$

$\therefore$ Coefficient of x^n

$x^n=b^n+ab^{n-1}+a^2b^{n-2}+.......+a^{n-1}b+a^n$

{which is a G.P. with $r=\frac{a}{b}$

$$\therefore \text{Its sum is } = \frac{b^n\left[1-\left(\frac{a}{b}\right)^{n+1}\right]}{1-\frac{a}{b}}\} = \frac{b^{n+1}-a^{n+1}}{b-a}$$

$$\therefore a_n=\frac{b^{n+1}-a^{n+1}}{b-a}$$

16. **(d)** $(1-y)^m(1+y)^n$

$=[1-{}^mC_1y+{}^mC_2y^2-......]$

$[1+{}^nC_1y+{}^nC_2y^2+.....]$

$$=1+(n-m)+\left\{\frac{m(m-1)}{2}+\frac{n(n-1)}{2}-mn\right\}y^2+.....$$

$\therefore a_1=n-m=10$

and $a_2=\dfrac{m^2+n^2-m-n-2mn}{2}=10$

So, $n-m=10$ and $(m-n)^2-(m+n)=20$

$\Rightarrow m+n=80$

$\therefore m=35, n=45$

17. **(b)** $T_{r+1}=(-1)^r.{}^nC_r(a)^{n-r}.(b)^r$ is an expansion of $(a-b)^n$

$\therefore$ 5th term $=t_5=t_{4+1}$

$=(-1)^4.{}^nC_4(a)^{n-4}.(b)^4={}^nC_4\,.\,a^{n-4}\,.\,b^4$

6th term $=t_6=t_{5+1}=(-1)^5\,{}^nC_5\,(a)^{n-5}\,(b)^5$

Given $t_5+t_6=0$

$\therefore {}^nC_4\,.\,a^{n-4}\,.\,b^4+(-{}^nC_5\,.\,a^{n-5}\,.\,b^5)=0$

$$\Rightarrow \frac{n!}{4!(n-4)!}\cdot\frac{a^n}{a^4}.b^4-\frac{n!}{5!(n-5)!}\cdot\frac{a^nb^5}{a^5}=0$$

$$\Rightarrow \frac{n!.a^nb^4}{4!(n-5)!.a^4}\left[\frac{1}{(n-4)}-\frac{b}{5.a}\right]=0$$

or, $\dfrac{1}{n-4}-\dfrac{b}{5a}=0 \Rightarrow \dfrac{a}{b}=\dfrac{n-4}{5}$

18. **(d)** We know that, $(1+x)^{20}={}^{20}C_0+{}^{20}C_1x+{}^{20}C_2x^2+......$ ${}^{20}C_{10}x^{10}+.....\,{}^{20}C_{20}x^{20}$

Put $x=-1$, $(0)={}^{20}C_0-{}^{20}C_1+{}^{20}C_2-{}^{20}C_3+......+{}^{20}C_{10}$ $-{}^{20}C_{11}....+{}^{20}C_{20}$

$\Rightarrow 0=2[{}^{20}C_0-{}^{20}C_1+{}^{20}C_2-{}^{20}C_3+.....-{}^{20}C_9]+{}^{20}C_{10}$

$\Rightarrow {}^{20}C_{10}=2[{}^{20}C_0-{}^{20}C_1+{}^{20}C_2-{}^{20}C_3+......-{}^{20}C_9+{}^{20}C_{10}]$

$\Rightarrow {}^{20}C_0-{}^{20}C_1+{}^{20}C_2-{}^{20}C_3+....+{}^{20}C_{10}=\frac{1}{2}\,{}^{20}C_{10}$

19. **(b)** We have

$$\sum_{r=0}^{n}(r+1)\,{}^nC_rx^r=\sum_{r=0}^{n}r.{}^nC_rx^r+\sum_{r=0}^{n}{}^nC_rx^r$$

$= \sum_{r=1}^{n} r \cdot \frac{n}{r} \, {}^{n-1}C_{r-1} x^r + (1+x)^n$

$= nx \sum_{r=1}^{n} {}^{n-1}C_{r-1} x^{r-1} + (1+x)^n$

$= nx(1+x)^{n-1} + (1+x)^n =$ RHS

∴ Statement 2 is correct.

Putting $x = 1$, we get

$\sum_{r=0}^{n} (r+1)^n C_r = n \cdot 2^{n-1} + 2^n = (n+2) \cdot 2^{n-1}$.

∴ Statement 1 is also true and statement 2 is a correct explanation for statement 1.

20. (a) $(8)^{2n} - (62)^{2n+1}$

$= (64)^n - (62)^{2n+1} = (63+1)^n - (63-1)^{2n+1}$

$= \left[{}^nC_0 (63)^n + {}^nC_1 (63)^{n-1} + {}^nC_2 (63)^{n-2} + \ldots\ldots + {}^nC_{n-1}(63) + {}^nC_n \right]$

$= \left[{}^{2n+1}C_0 (63)^{2n+1} - {}^{2n+1}C_1 (63)^{2n} + {}^{2n+1}C_2 (63)^{2n-1} - \ldots\ldots + (-1)^{2n+1} \, {}^{2n+1}C_{2+1} \right]$

$= 63 \times \left[{}^nC_0 (63)^{n-1} + {}^nC_1 (63)^{n-2} + {}^nC_2 (63)^{n-3} + \ldots\ldots \right] + 1$

$- 63 \times \left[{}^{2n+1}C_0 (63)^{2n} - {}^{2n+1}C_1 (63)^{2n-1} + \ldots\ldots \right] + 1$

⇒ 63 × some integral value + 2

⇒ $8^{2n} - (62)^{2n+1}$ when divided by 9 leaves 2 as the remainder.

21. (b) $S_2 = \sum_{j=1}^{10} j \, {}^{10}C_j = \sum_{j=1}^{10} 10 \, {}^9C_{j-1}$

$= 10\left[{}^9C_0 + {}^9C_1 + {}^9C_2 + \ldots + {}^9C_9 \right] = 10 \cdot 2^9$

22. (b) $(1 - x - x^2 + x^3)^6 = [(1-x) - x^2(1-x)]^6$

$= (1-x)^6 (1-x^2)^6$

$= (1 - 6x + 15x^2 - 20x^3 + 15x^4 - 6x^5 + x^6)$

$\times (1 - 6x^2 + 15x^4 - 20x^6 + 15x^8 - 6x^{10} + x^{12})$

Coefficient of $x^7 = (-6)(-20) + (-20)(15) + (-6)(-6)$

$= -144$

23. (a) $(\sqrt{3}+1)^{2n} - (\sqrt{3}-1)^{2n}$

$= \left[(\sqrt{3}+1)^2\right]^n - \left[(\sqrt{3}-1)^2\right]^n$

$= (4+2\sqrt{3})^n - (4-2\sqrt{3})^n$

$= 2^n\left[(2+\sqrt{3})^n - (2-\sqrt{3})^n\right]$

$= 2^n \times 2\left[{}^nC_1 \, 2^{n-1}\sqrt{3} + {}^nC_3 \cdot 2^{n-3} 3\sqrt{3} + \ldots \right]$

$= 2^{n+1}\sqrt{3}\left[{}^nC_1 \cdot 2^{n-1} + {}^nC_3 2^{n-3} \cdot 3 + \ldots \right]$

$= \sqrt{3} \times$ Some integer ∴ irrational number

24. (c) Given expression can be written as

$\left((x^{1/3}+1) - \left(\frac{\sqrt{x}+1}{\sqrt{x}} \right) \right)^{10} = \left(x^{1/3} + 1 - 1 - \frac{1}{\sqrt{x}} \right)^{10}$

$= (x^{1/3} - x^{-1/2})^{10}$

General term $= T_{r+1} = {}^{10}C_r (x^{1/3})^{10-r} (-x^{-1/2})^r$

$= {}^{10}C_r \, x^{\frac{10-r}{3}} \cdot (-1)^r \cdot x^{-\frac{r}{2}} = {}^{10}C_r (-1)^r \cdot x^{\frac{10-r}{3} - \frac{r}{2}}$

Term will be independent of x when $\frac{10-r}{3} - \frac{r}{2} = 0$

⇒ $r = 4$

So, required term $= T_5 = {}^{10}C_4 = 210$

25. (b) Consider $(1 + ax + bx^2)(1 - 2x)^{18}$

$= (1 + ax + bx^2)[{}^{18}C_0 - {}^{18}C_1(2x) + {}^{18}C_2(2x)^2 - {}^{18}C_3(2x)^3 + {}^{18}C_4(2x)^4 - \ldots\ldots]$

Coeff of $x^3 = {}^{18}C_3(-2)^3 + a.(-2)^2 . {}^{18}C_2 + b(-2) . {}^{18}C_1 = 0$

Coeff. of $x^3 = -{}^{18}C_3 . 8 + a \times 4 . {}^{18}C_2 - 2b \times 18 = 0$

$= -\frac{18 \times 17 \times 16}{6} . 8 + \frac{4a + 18 \times 17}{2} - 36b = 0$

$= -51 \times 16 \times 8 + a \times 36 \times 17 - 36b = 0$

$= -34 \times 16 + 51a - 3b = 0$

$= 51a - 3b = 34 \times 16 = 544$

$= 51a - 3b = 544$ ....(i)

Only option number (b) satisfies the equation number (i)

26. (c) $(1 - 2\sqrt{x})^{50} = {}^{50}C_0 - {}^{50}C_1 \, 2\sqrt{x} + {}^{50}C_2 (2\sqrt{x})^2$ ...(1)

$(1 + 2\sqrt{x})^{50} = {}^{50}C_0 + {}^{50}C_1 \, 2\sqrt{x} - {}^{50}C_2 (2\sqrt{x})^2 + \ldots + {}^{50}C_3 (2\sqrt{x})^3 - {}^{50}C_4 (2\sqrt{x})^4$ ...(2)

Adding equation (1) and (2)

$(1 - 2\sqrt{x})^{50} + (1 + 2\sqrt{x})^{50}$

$= 2\left[{}^{50}C_0 + {}^{50}C_2 2^2 x + {}^{50}C_4 2^3 x^2 + \ldots \right]$

Putting x = 1, we get above as $\frac{3^{50}+1}{2}$

27. (b) Total number of terms $= {}^{n+2}C_2 = 28$

$(n+2)(n+1) = 56$

$x = 6$

Sum of coefficients $= (1 - 2 + 4)^n = 3^6 = 729$